1. Does there exists a solution of the Cauchy problem

\[ yu_x = xu_y \quad \text{in } \mathbb{R}^2, \quad u \big|_{\Gamma} = \cos y, \quad \Gamma = \{(1, y) : y \in \mathbb{R}\} \]

in a neighborhood of the point \((1, 0)\)? Is the solution unique?
2. Let $\Omega \subset \mathbb{R}^n$ be an unbounded open set and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in $\Omega$. Show that if

$$\lim_{|x| \to \infty, x \in \Omega} u(x) = 0$$

then

$$\sup_{\Omega} |u| = \sup_{\partial \Omega} |u|.$$ 

[Hint. Consider the open sets $\Omega_R = \Omega \cap B_R$ and let $R \to \infty$.]
3. Let $u \in C^2(\mathbb{R}^n \times (0,1]) \cap C(\mathbb{R}^n \times [0,1])$ be a bounded solution of the initial value problem for the heat equation with nonnegative initial data:

$$\Delta u - u_t = 0 \text{ in } \mathbb{R}^n \times (0,1], \quad u(\cdot,0) = g \geq 0.$$ 

Prove that there exists a constant $C_n > 0$, depending only on the dimension $n$ such that

$$\sup_{|x| \leq 1} u(x, \frac{1}{2}) \leq C_n \inf_{|z| \leq 1} u(z,1)$$

[Hint: You may want to use the inequality $|x - y|^2 \geq \frac{1}{2} |z - y|^2 - |x - z|^2$ for $x, y, z \in \mathbb{R}^n$.]
4. (a) Let \( u \) be a harmonic function in \( B_R := \{ x \in \mathbb{R}^n : |x| < R \} \) and \( f \) a \( C^\infty \) radial function (i.e., \( f(y) = f(|y|) \)) supported in \( B_R \). Show that
\[
\int_{B_R} u(y) f(y) dy = Mu(0), \quad \text{where} \quad M := \int_{\mathbb{R}^n} f(y) dy.
\]
[Hint. Use the spherical mean value property.]

(b) Let \( f \) be as above and consider the Newtonian potential
\[
U(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y)f(y)dy, \quad x \in \mathbb{R}^n,
\]
where \( \Phi \) is the fundamental solution of the Laplace equation in \( \mathbb{R}^n \). Prove that
\[
u(x) = M\Phi(x), \quad \text{for any} \quad x \in \mathbb{R}^n \setminus B_R.
\]
5. Let $u \in C^2(\mathbb{R}^n \times [0, \infty))$ be a solution of the initial value problem

\begin{align*}
    u_{tt} - \Delta u &= 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\
    u(x, 0) &= 0, \quad u_t(x, 0) = h(x),
\end{align*}

in space dimensions $n = 1, 2, \text{or } 3$. Suppose that $|h(x)| \leq M$ for all $x \in \mathbb{R}^n$.

(a) Prove that $|u(x, t)| \leq M t$, for all $x \in \mathbb{R}^n$, $t > 0$.

(b) Show that in general it is not be true that $|u_t(x, t)| \leq M$ for all $x \in \mathbb{R}^n$, $t > 0$.

[Hint: Consider a radially symmetric $h(x) = h(|x|)$ and explicitly compute $u_t(0, t)$.]