**1.** Does there exists a solution of the Cauchy problem [20pt]

$$yu_x = xu_y \quad \text{in } \mathbb{R}^2, \quad u\big|_\Gamma = \cos y, \quad \Gamma = \{(1,y) : y \in \mathbb{R}\}$$

in a neighborhood of the point $(1,0)$? Is the solution unique?

**2.** Let $\Omega \subset \mathbb{R}^n$ be an *unbounded* open set and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in $\Omega$. Show that if [20pt]

$$\lim_{\substack{|x|\to\infty \\ x\in\Omega}} u(x) = 0$$

then

$$\sup_{\overline{\Omega}} |u| = \sup_{\partial\Omega} |u|.$$

[*Hint.* Consider the open sets $\Omega_R = \Omega \cap B_R$ and let $R \to \infty$.]

**3.** Let $u \in C^2(\mathbb{R}^n \times (0,1]) \cap C(\mathbb{R}^n \times [0,1])$ be a *bounded* solution of the initial value problem for the heat equation with *nonnegative* initial data: [20pt]

$$\Delta u - u_t = 0 \quad \text{in } \mathbb{R}^n \times (0,1], \quad u(\cdot, 0) = g \geq 0.$$

Prove that there exists a constant $C_n > 0$, depending only on the dimension $n$ such that

$$\sup_{|x|\leq 1} u(x, \tfrac{1}{2}) \leq C_n \inf_{|z|\leq 1} u(z, 1)$$

[*Hint:* You may want to use the inequality $|x-y|^2 \geq \frac{1}{2}|z-y|^2 - |x-z|^2$ for $x, y, z \in \mathbb{R}^n$.]

**4.** (a) Let $u$ be a harmonic function in $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ and $f$ a $C^\infty$ radial function (i.e., $f(y) = f(|y|)$) supported in $B_R$. Show that [20pt]

$$\int_{B_R} u(y)f(y)dy = Mu(0), \quad \text{where } M := \int_{\mathbb{R}^n} f(y)dy.$$

[Hint. Use the spherical mean value property.]

(b) Let $f$ be as above and consider the Newtonian potential

$$U(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy, \quad x \in \mathbb{R}^n,$$

where $\Phi$ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$. Prove that

$$u(x) = M\Phi(x), \quad \text{for any} \quad x \in \mathbb{R}^n \setminus B_R.$$

**5.** Let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ be a solution of the initial value problem [20pt]

$$u_{tt} - \Delta u = 0, \quad x \in \mathbb{R}^n,\ t > 0,$$
$$u(x,0) = 0, \quad u_t(x,0) = h(x),$$

in space dimensions $n = 1$, 2, or 3. Suppose that $|h(x)| \leq M$ for all $x \in \mathbb{R}^n$.

(a) Prove that $|u(x,t)| \leq Mt$, for all $x \in \mathbb{R}^n$, $t > 0$.

(b) Show that in general it is not be true that $|u_t(x,t)| \leq M$ for all $x \in \mathbb{R}^n$, $t > 0$.
[*Hint:* Consider a radially symmetric $h(x) = h(|x|)$ and explicitly compute $u_t(0,t)$.]