QUALIFYING EXAMINATION
JANUARY 1994
MATH 530

Please answer each question on a separate sheet of paper!

1. (a) The function $f(z) = \dfrac{4}{(1+z)(3-z)}$ has Laurent series

   (I) $\sum_{k=0}^{\infty} \left(1 + \frac{(-1)^k}{3^{k+1}}\right)(z-2)^k$ (II) $\sum_{k=-\infty}^{-1} \left(-1 + (-3)^{-(k+1)}\right)(z-2)^k$

   (III) $\sum_{k=-\infty}^{-1} -(z-2)^k + \sum_{k=0}^{\infty} \frac{(-1)^k}{3^{k+1}}(z-2)^k$

   Find the sets of absolute convergence for each of these series.

   (b) Suppose the function f is analytic in the plane except for simple poles at $z = -1$ and $z = 3$ and has Laurent series

   (I) $\displaystyle\sum_{k=0}^{\infty} a_k(z-2)^k$ (II) $\displaystyle\sum_{k=-\infty}^{-1} b_k(z-2)^k$ (III) $\displaystyle\sum_{k=-\infty}^{\infty} c_k(z-2)^k$

   Letting $\Gamma = \{z : |z-3| = 1\}$ oriented counterclockwise, express the integral

   $$\int_\Gamma f(z)\,dz$$

   in terms of the coefficients of the Laurent series above, and justify your answer.

2. The function g is analytic in the plane except for four poles, including poles at -1, 2, and $3+4i$. Moreover, g is real–valued on the interval $\{z : \mathrm{Im}(z) = 0,\ -1 < z < 2\}$ of the real axis.

   (a) Prove that g is real–valued on the whole real axis except for its poles.

   (b) Find the location of the fourth pole and justify your answer.

3. Suppose there is $R > 1$ so that $h(z)$ is analytic in the disk $\{z : |z| < R\}$.
   Prove that if $|h(z)| \leq 1$ for $|z| \leq 1$ and $h(0) = 0$ and $h(1) = 1$, then $|h'(1)| \geq 1$.
   (Hint: you may wish to consider $\lim_{r\to 1^-}(h(1) - h(r))/(1-r)$.)

4. (a) Express the arctangent function in terms of the logarithm.

   (b) Let $A(z)$ be the branch of the arctangent function that is analytic except for $\{z : \mathrm{Re}(z) = 0,\ |\mathrm{Im}(z)| \geq 1\}$ and that has $A(0) = \pi$. Find, justifying your work,

   $$\lim_{t\to 0^+} \mathrm{Re}\,(A(t+i))$$

   (where, as usual, "$t \to 0^+$" means t is positive and real as it approaches 0).

5. Use the residue theorem to evaluate

   $$\int_0^\infty \frac{t}{4+t^4}\,dt$$

   Justify your answer by careful statements of your contours and estimates.