QUALIFYING EXAMINATION

August 1999

MATH 530 - Prof. Bell

Notation: $D_r(a)$ denotes the disk, $\{z \in \mathbb{C} : |z - a| < r\}$.

1. *(20 pts)* Find the smallest integer n such that there is no $z \in \mathbb{C}$ with

$$z^{11} + z^5 + 8z + 1999 = 0$$

and $|z| \geq n$. Explain. ($3^{11} = 177147$, $3^5 = 243$, $2^{11} = 2048$, $2^5 = 32$.)

2. *(20 pts)* Let $f : \mathbb{C} - \{0, 1\} \to \mathbb{C}$ be an analytic function such that $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ for $|z| > 1$, where $a_n = 1$ for $n < 0$ and $a_n = 1/n!$ for $n \geq 0$. Determine what type of singularity f has at 0, 1 and ∞.

3. *(20 pts)* Assume that F is a one-to-one analytic mapping of the square

$$\{z \,:\, -1 < \text{Re } z < 1, -1 < \text{Im } z < 1\}$$

onto the unit disk such that $F(0) = 0$. Prove that $F(iz) = iF(z)$ for all z.

4. *(20 pts)* Let $f(z) = \sum_{n=0}^{\infty} z^{n!}$. Show that the radius of convergence of this power series is one. Let u denote a root of unity. Show that

$$\lim_{r \to 1-} f(ru) = \infty.$$

Let $\Omega = D_1(0) \cup D_\epsilon(1)$. Is there an $\epsilon > 0$ and a meromorphic function F on Ω such that $F = f$ on the unit disk? Explain.

5. *(20 pts)* Let $\mathcal{F}$ denote the family of all analytic maps f of the unit disk to itself for which $f(1/2) = 0$. Find

$$\sup_{f \in \mathcal{F}} \{\text{Im } f(0)\}.$$

Explain.