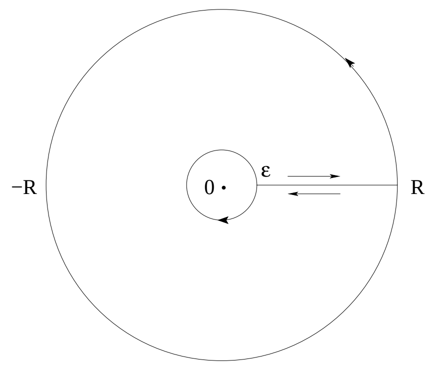
MATH 530 Qualifying Exam

August 2001

1. *(20 pts)* Suppose that $f(z)$ is analytic in the unit disc $D_1(0)$ with $f(0) = 0$ and $|f(z)| < 1$ on $D_1(0)$. Prove that $\sum_{n=1}^{\infty} f(z^n)$ converges to a function which is also analytic in $D_1(0)$. Be sure to carefully explain every step in your proof and to write out the statement of any theorems you refer to.

2. *(20 pts)* Evaluate the integral $\displaystyle\int_0^{\infty} \frac{\ln x}{(x+2)^2}\, dx$. Hint: Define a branch of $\log z$ and integrate $\dfrac{(\log z)^2}{(z+2)^2}$ around an appropriately chosen contour (see figure), etc.



3. *(20 pts)* Suppose that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic in $\{z : |z| < R\}$. Prove that the series

$$\phi(z) := \sum_{n=0}^{\infty} \frac{c_n z^n}{n!}$$

converges on the whole complex plane. For any fixed r with $0 < r < R$, prove that there is a positive constant M (which might depend on r) such that the estimate

$$|\phi(z)| < M e^{|z|/r}$$

holds for all $z \in \mathbb{C}$.

4. *(20 pts)* Let $\mathcal{S}$ denote the strip

$$\mathcal{S} = \left\{ z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2} \right\}.$$

Show that the entire function e^{e^z} is bounded on the boundary of $\mathcal{S} \subset \mathbb{C}$, but it is not bounded on $\mathcal{S}$.

5. *(20 pts)* Suppose that $f(z)$ is continuous on $\{z : |z| \leq 1\} - \{1\}$ and analytic on $\{z : |z| < 1\}$. Prove that if $f(z)$ is *bounded* on $\{z : |z| < 1\}$, then

$$|f(z)| \leq \sup_{|\zeta|=1, \zeta \neq 1} |f(\zeta)|$$

for each z in $\{z : |z| < 1\}$. Is this inequality valid if f is not assumed to be bounded?