Qualifying Exam **August 2003**
MA 530 **Professor Donnelly**

I. The first four questions are each worth ten points.

1. Write $(1 - i\sqrt{3})^{85}$ in the form $a + ib$.

2. Let Γ be the contour consisting of two circles $|z-1| = 2$, counterclockwise, and $|z-1| = 1$, clockwise. Find the value of the winding number in each component of the complement of Γ.

3. How many zeroes does $2z^2 - e^{z/2}$ have in $|z| < 1$?.

4. Find the first five nonzero terms in the Taylor series of $\dfrac{\sin z}{z^2+1}$, about $z = 0$.

II. The remaining four questions are each worth fifteen points.

5. Give an example of a harmonic function, which is not the real part of a holomorphic function. Explain why your answer is correct.

6. Suppose f is a holomorphic mapping from the unit disc to itself. Show that, for all $|a| < 1$,
$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}$$

7. If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity, in the sense that u becomes harmonic in $|z| < \rho$, when $u(0)$ is properly defined.

8. By making the substitution $x = iy$ and noticing that $y \to \infty$ as $x \to \infty$, we formally transform the integral $\displaystyle\int_0^\infty \frac{x\,dx}{x^4+4}$ into the integral $-\displaystyle\int_0^\infty \frac{y\,dy}{y^4+4}$, which is the negative of the original integral. Hence, it appears that the value of the integral is zero. But the true value of the integral is found by elementary methods to be $\pi/8$. What is the fallacy? Explain using complex analysis.