QUALIFYING EXAMINATION

JANUARY 2005

MATH 530 - Prof. Lempert

Each problem is worth 5 points.

1. Suppose that f is holomorphic and nonconstant in a disc $|z-a|<r$. Show that

$$\lim_{z\to a}\frac{\log|f(z)-f(a)|}{\log|z-a|}$$

exists, and is a nonnegative integer.

2. Prove that if g has a pole and h has an essential singularity at c then gh has an essential singularity at c.

3. Evaluate

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+1)(x^2+4)}dx$$

(and show your work).

4. Show that if a holomorphic function $\phi : \mathrm{C}\to\mathrm{C}$ satisfies $|\phi(z)|\le e^{|z|}$ for all z, then there is a $c>0$ such that $|\phi'(z)|\le ce^{|z|}$ for all z.

5. Construct a one–to–one holomorphic map from the half–disc

$$D=\{z\,:\,|z|<1,\,\mathrm{Im}\,z>0\}$$

onto the unit disc U (i.e., a biholomorphic map $D\to U$).

6. Let $P(z)=z^n+a_1z^{n-1}+\cdots+a_n$ be a polynomial and $0<\theta<\pi/(2n)$. Show that

$$\begin{aligned} e^{P(z)}&\to\infty, &&\text{as } z\to\infty, &&|\arg z|<\theta, &&\text{and}\\ e^{P(z)}&\to 0, &&\text{as } z\to\infty, &&|\arg z-\pi/n|<\theta. \end{aligned}$$

7. Prove that if Q is any polynomial, then

$$|Q(z)-\frac{1}{z}|\ge 1$$

for some z with $|z|=1$.