MATH 530 Qualifying Exam

August 2006

1. *(10 pts)* Let f be analytic in the punctured disk $D' = \{z : 0 < |z - z_0| < r\}$. Prove that if

$$|f''(z)| \leq \frac{1}{|z - z_0|^2}$$

for all $z \in D'$, then f has a removable singularity at z_0.

2. *(20 pts)* Adapt the proof of Schwarz's lemma to prove the following. Let f be analytic in the unit disk $U = \{z : |z| < 1\}$ having zeros at $z = 1/2$ and $z = -1/2$, and such that $|f(z)| < 1$ for all $z \in U$. Prove that $|f(i/2)| \leq 8/17$.

3. i) *(10 pts)* Suppose that the functions $f_1, f_2, f_3, \ldots$ are analytic in a domain Ω, and the sequence converges uniformly on all compact subsets of Ω to a function f. Prove that the sum $\sum_{n=1}^{\infty} \frac{f_n(z)}{2^n}$ converges to a function g which is analytic on Ω.

ii) *(10 pts)* Suppose that the functions in i) satisfy the condition

$$|f_n(z)| \leq 10, \quad n = 1, 2, \ldots$$

throughout Ω, and recall that $g(z)$ is the sum in i). Prove that if $g(z_0) = 10$ for some $z_0 \in \Omega$, then $g(z) \equiv 10$ in Ω.

4. i) *(10 pts)* Determine the set of points in the complex plane where $\sin z$ is pure real.

ii) *(10 pts)* Determine the set of points in the complex plane where $|e^{z^2}| > 1$.

5. *(10 pts)* Let $C = \{z : |z| = 5\}$, traversed once counterclockwise. Compute the contour integral

$$\int_C \frac{e^{1/z} + \sin z}{z}\, dz.$$

6. *(10 pts)* Let f be analytic in a domain which contains the closed disk $\{z : |z| \leq 2\}$, and let C be its boundary $\{z : |z| = 2\}$ traversed once counterclockwise.

Prove that $\mathrm{Re} f(0) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{\mathrm{Re} f(z)}{z}\, dz.$

7. *(10 pts)* Evaluate

$$\int_{1-i\infty}^{1+i\infty} \frac{e^z}{1+z^2}\, dz,$$

i.e., along the line $\{z : \mathrm{Re}\ z = 1\}$ in the direction of increasing imaginary part.