Math 530
Qualifying Examination
August 15, 2007 – Profs. Drasin and Weitsman

(Please write on one side of the paper. Show work and details.)

1.(a) Find two conjugate functions to the harmonic function $u(z) = \log|z|$ with domain $\{|z-1| < 1\}$.

(b) Show there is no conjugate function to $u(z) = \log|z|$ in the punctured disk $\{0 < |z| < 1\}$.

2. Find all conformal self–maps of $\{0 < |z| < 1\}$.

3. Let f be analytic inside the square D having vertices at $(-2, \pm 2)$ and $(2, \pm 2)$, and suppose f is continuous on ∂D. Suppose $\Re e(f(z)) = 0$ at precisely four points of ∂D. Show that f has *at most* two zeros in D. (This is only an upper bound; use the argument principle.)

4. Compute

$$\int_0^\infty \frac{\log x}{x^2+4}\,dx$$

by residues.

5. Let $f(z) = \sum_0^\infty a_n z^n, g(z) = \sum_0^\infty b_n z^n$ be analytic in $U : \{|z| < 1\}$ and continuous on ∂U, oriented counterclockwise. Prove that

$$\frac{1}{2\pi i}\int_{\partial U} f(\zeta)g(z/\zeta)\frac{d\zeta}{\zeta} = \sum_0^\infty a_n b_n z^n \qquad (z \in U)$$

(You must justify all steps in your argument.)

6. Let R be the boundary of the rectangle with vertices $(-3, \pm 2), (+3, \pm 2)$, and let

$$F(z) = \int_R z e^{2|\zeta|^2}\frac{d\zeta}{(\zeta - z)^2}.$$

(a) Prove that F is analytic inside R.

(b) Try to obtain as good an upper bound for $|F''(0)|$ as possible.