MATH 530 Qualifying Exam

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Notation: $D_r(a) = \{z : |z - a| < r\}$.

1. *(20 pts)* i) Let t represent a non-zero real number. Find a linear fractional transformation $T(z)$ such that $T(0) = -i$, $T(t) = 1$, and $T(\infty) = i$.
ii) For which values of t does T map the upper half plane onto the unit disc $D_1(0)$? Explain.

2. *(15 pts)* Suppose that f is analytic in a disc $D_r(0) = \{z : |z| < r\}$, $r > 0$. Prove that f is an even function (i.e., $f(z) = f(-z)$) for $z \in D_r(0)$, if and only if the power series for f centered at the origin has only even powers.

3. *(10 pts)* Evaluate
$$\int_{-\infty}^{\infty} \frac{\cos t}{1 + t^4}\, dt.$$

4. *(20 pts)* Suppose that f is analytic in the disc $\mathcal{D} = D_2(0)$ and continuous on its closure $\overline{\mathcal{D}}$. Prove that if $|f(z)| \le |\sin z|$ for all z in the boundary of $\mathcal{D}$, then $|f(\frac{\pi}{2})| \le 4/\pi$.

5. *(20 pts)* Suppose the power series $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence R_1 with $0 < R_1 < \infty$, and the power series $\sum_{n=0}^{\infty} b_n z^n$ has a has a radius of convergence R_2 with $0 < R_2 < \infty$. Suppose further that $b_n \neq 0$ for all n. Prove that the power series
$$\sum_{n=0}^{\infty} \frac{a_n}{b_n} z^n$$
has a radius of convergence R_3 satisfying
$$R_3 \le \frac{R_1}{R_2}.$$

6. *(15 pts)* Suppose $\mathcal{A}$ is a finite set of points in the complex plane and let $\star$ denote the union of the closed line segments joining each $a \in \mathcal{A}$ to the origin. If f is analytic on $\mathbb{C} - \mathcal{A}$ and is such that
$$0 = \sum_{a \in \mathcal{A}} \mathrm{Res}_a f,$$
prove that f has an analytic anti-derivative on $\mathbb{C} - \star$.