MATH 530 Qualifying exam

January 2009 (A. Eremenko)

Each problem is worth 10 points.
You can use any theorem proved in class or a theorem from the textbook if you state it completely and correctly.

1. Let f_n be a sequence of injective analytic functions in the open unit disc, and suppose that

$$f = \lim_{n \to \infty} f_n$$

uniformly on compact subsets of the unit disc. Prove that f is either injective or constant.

2. Let Ω be a bounded region in the plane and $f : \Omega \to \Omega$ an analytic function that maps Ω into itself. Suppose that there exists a point $z_0 \in \Omega$ such that $f(z_0) = z_0$. Prove that $|f'(z_0)| \leq 1$.

Hint: *If a function maps something into itself, iterating it is a good idea.*

3. Consider the Taylor expansion in a neighborhood of the point i:

$$z \cot z = \sum_{n=0}^{\infty} c_n (z - i)^n.$$

What is the radius of convergence of the series in the right hand side?

4. Let f be an analytic function in the strip

$$\Pi = \{x + iy : |x| < \pi/4, \ -\infty < y < \infty\},$$

and suppose that $|f(z)| \leq 1$ and $f(0) = 0$. Prove that $|f(z)| \leq |\tan z|$ for all $z \in \Pi$.

5. Find a harmonic function in the region $\{z : |z| < 1, \ \operatorname{Im} z > 0\}$ whose boundary values are 1 on the interval $(-1, 1)$ and 0 on the half-circle.

6. Let f and g be two analytic functions in some region D, and suppose that $f(z) + \overline{g(z)}$ is real for all $z \in D$. Prove that $f - g$ is constant.

7. For all real a, evaluate

$$\int_0^{\pi} \tan(x + ia) dx.$$

Use the principal value when $a = 0$.

8. Let f be a meromorphic function in a neighborhood of 0 and ϕ an analytic function in a neighborhood of 0 with the properties $\phi(0) = 0$ and $\phi'(0) \neq 0$. Prove that

$$\operatorname{res}_0 f = \operatorname{res}_0[(f \circ \phi)\phi'].$$