MATH 530 Qualifying Exam

August 2010 (S. Bell, A. Eremenko)

Each problem is worth 20 points

1. Compute $\int_0^\infty \frac{x \sin x}{1+x^2}\, dx$. *Hint:* Use a rectangular contour and let the dimensions go to infinity one at a time. If you claim that a certain term goes to zero, prove that it does.

2. Find a one-to-one conformal map from the strip $\{z : 0 < \text{Im } z < 1\}$ onto the half-strip $\{z : 0 < \text{Re } z,\ 0 < \text{Im } z < 1\}$. (You may express your answer as a composition of more elementary maps.)

3. Prove that the function $f(z) = 1/z$ does not have an analytic antiderivative on $\mathbb{C} - \{0\}$. Find all integers $n = 0, \pm 1, \pm 2, \ldots$ such that the function $g(z) = z^n e^{1/z}$ has an analytic antiderivative on $\mathbb{C} - \{0\}$.

4. Find all real valued harmonic functions on the plane that are constant on all vertical lines.

5. It is a fact that, if $n \in \mathbb{Z}$, then

$$\frac{1}{\sin^2 z} - \frac{1}{(z-\pi n)^2}$$

has a removable singularity at $z = \pi n$.

a) Demonstrate this fact in case $n = 0$.

b) Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-\pi n)^2}$$

converges uniformly on every bounded set after dropping finitely many terms.

c) Finally, use Liouville's Theorem to prove that

$$\frac{1}{\sin^2 z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-\pi n)^2}.$$