MATH 530 Qualifying Exam

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Note: $\mathbb{D}$ is the open unit disk.

1. Define $f(z) = \displaystyle\int_0^1 \frac{dt}{1+tz}$.

   (a) (10 points) Use Morera's Theorem to show that f is analytic in $\mathbb{D}$.

   (b) (10 points) Find a power series expansion for $f(z)$ on $\mathbb{D}$.

2. (20 points) Suppose f is entire and that there is some $K > 0$ so that if $|z| \geq K$, then

$$|\mathrm{Re}\ f(z)| \geq |\mathrm{Im}\ f(z)|.$$

Prove that f is constant.

3. (20 points) Suppose f is a holomorphic function (not necessarily bounded) on $\mathbb{D}$ such that $f(0) = 0$. Prove that the infinite series $\sum_{n=1}^{\infty} f(z^n)$ converges uniformly on compact subsets of $\mathbb{D}$.

4. (20 points) Define a family $\mathcal{F}$ of functions holomorphic in an open set Ω to be a normal family if every sequence from $\mathcal{F}$ contains either a subsequence that converges uniformly on every compact set $K \subset \Omega$ or a subsequence that tends uniformly to ∞ on every compact set.

   Let f be entire, and let $\mathcal{F}$ be the family $\{f(kz) : k \in \mathbb{C}\}$. Also, let Ω be the annulus $1/2 < |z| < 2$. Prove that $\mathcal{F}$ is a normal family in Ω if and only if f is a polynomial.

5. Let Ω be a bounded, open, connected set, and suppose that $f_1, f_2, \ldots, f_n$ are holomorphic in Ω and continuous on the closure of Ω. Let $g = |f_1| + |f_2| + \cdots + |f_n|$.

   (a) (10 points) Prove that the maximum of g is attained on the boundary of Ω.

   (b) (10 points) Prove that if g is constant, then each f_k is also constant.