MATH 530 Qualifying Exam

August 2013 (S. Bell)

Each problem is worth 20 points

1. Suppose Ω is a domain in the complex plane and $F(z,t)$ is a continuous function on $\Omega \times I$ where $I = [0,1]$ is the unit interval in $\mathbb{R}$. Suppose further that $F(z,t)$ is analytic in z on Ω for each fixed $t \in I$. Prove that

$$g(z) = \int_0^1 F(z,t)\ dt$$

is analytic on Ω. What can be said if $F(z,t)$ is only assumed to be analytic in $z \in \Omega$ for all *rational* values of t (when held fixed) in I.

2. Let C_1 denote the unit circle parametrized in the standard sense. Compute

$$\int_{C_1} \frac{1}{z^2 + z - \sigma}\ dz$$

where σ is a real number satisfying $0 < \sigma < 2$.

3. Suppose that $f(z)$ is analytic on the upper half plane and maps the upper half plane into the unit disc. Prove that $|f'(i)| \leq \frac{1}{2}$. What can be said if $|f'(i)| = \frac{1}{2}$?

4. Suppose f is analytic on a domain Ω and is not identically zero there. Let $Z_f = \{z \in \Omega : f(z) = 0\}$ denote the zero set of f. Prove that $\Omega - Z_f$ is connected. Is the same true if f is assumed to be a real valued harmonic function?

5. Suppose a_n is a sequence of distinct non-zero complex numbers such that

$$\sum_{n=1}^{\infty} |a_n|^{-1} < \infty.$$

Let $\mathcal{A} = \{a_n : n = 1, \ldots, \infty\}$.

a) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z - a_n}$ converges to a function $f(z)$ that is analytic on $\mathbb{C} - \mathcal{A}$.

b) For $z \in \mathbb{C} - \mathcal{A}$, let

$$G(z) = \exp\left(\int_{\gamma_0^z} f(w)\ dw \right)$$

where γ_0^z is a curve in $\mathbb{C} - \mathcal{A}$ that starts at the origin and ends at z. Prove that G is well defined and analytic on $\mathbb{C} - \mathcal{A}$. Show that G has removable singularities at each of the points a_n. Finally, show that the points a_n are in fact simple zeroes of G.