MATH 530 Qualifying Exam

January 2013 (S. Bell)

Each problem is worth 20 points

1. Convert the integral

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$$

into a contour integral of the form $\int_C f(z)\,dz$ where C is the unit circle and f is a rational function, and then use the Residue Theorem to compute the integral.

2. Find the radius of convergence about $z=0$ of the power series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} z^{2n}.$$

3. Find a one-to-one conformal map from the quarter disc

$$\{z = re^{i\theta} : 0 < r < 1, 0 < \theta < \pi/2\}$$

onto the unit disc. You may express your solution as a composition of simpler mappings.

4. Compute

$$\int_0^{\infty} \frac{1}{x^3+1}\,dx$$

by integrating $f(z) = 1/(z^3+1)$ around the contour that follows the real line from zero to R, then follows the circle Re^{it} from $t=0$ to $t=2\pi/3$, and then follows the line $te^{i2\pi/3}$ from $t=R$ back to $t=0$. Use the Residue Theorem and let $R\to\infty$.

5. Suppose that $f(z)$ is an analytic function that maps the unit disc into itself with two distinct fixed points, i.e., with points z_1 and z_2 in the unit disc, $z_1 \neq z_2$, such that $f(z_j) = z_j$ for $j = 1, 2$. Prove that $f(z) = z$ for all z.