MATH 530 Qualifying Exam

August 2014 (S. Bell)

Each problem is worth 20 points

1. State the Maximum Modulus Principle and Liouville's Theorem. Show that the Maximum Modulus Principle implies Liouville's Theorem.

2. Find a one-to-one conformal mapping from the unit disc minus the real interval $[0, 1)$ onto the unit disc.

3. a) Compute the residue at the origin of $\dfrac{e^z}{\sin^2 z}$. (Don't use a memorized formula. Compute it.)

b) Compute $\int_\gamma e^{1/z}\,dz$ where γ is a closed curve that avoids the origin, but can cross itself and wrap around the origin any number of times.

4. Suppose that f is a meromorphic function on a simply connected domain Ω. This means that there is a discrete set of points $\mathcal{A} \subset \Omega$ such that f is analytic on $\Omega - \mathcal{A}$ and f has poles at the points in $\mathcal{A}$. Prove that the following three conditions are equivalent:

i) $f = F'$ for a meromorphic function F on Ω,

ii) $0 = \int_\gamma f\,dz$ for every closed curve γ in $\Omega - \mathcal{A}$.

iii) The residue of f is zero at each point $a \in \mathcal{A}$.

5. Let f be a fixed entire function, and let $\mathcal{F}$ be the set of all functions of the form $g(z) = f(kz)$, where k runs over all complex constants. A family of analytic functions on a domain is called *normal* if every sequence of functions in the family contains either a subsequence that converges uniformly on compact subsets, or a subsequence that tends uniformly to ∞ on compact subsets.

Prove that if $\mathcal{F}$ is a normal family in the annulus $r < |z| < R$ where $0 < r < R$, then f must be a polynomial.