MA 53000 QUALIFIER, 8/13/2015

Each problem is worth 5 points. Make sure that you justify your answers. $D_R(a) \subset \mathbb{C}$ stands for the open disc of radius R centered at a. The set of holomorphic functions on an open $\Omega \subset \mathbb{C}$ is denoted $\mathcal{O}(\Omega)$.

Notes, books, crib sheets, and electronic devices are not allowed.

1. If we expand the function $1/\cos z$ in a Taylor series about 0, for what R will this series converge on $D_R(0)$? For what R will it converge uniformly on $D_R(0)$?

2. Let $c \in \mathbb{C}$ be given. Is there a holomorphic $f : \mathbb{C} \to \mathbb{C}$ such that $f(1/k) = c^k$ for all $k \in \mathbb{N}$?

3. Compute

$$\int_{|z|=4} \frac{dz}{z^2(e^z - 1)}.$$

4. Suppose $g \in \mathcal{O}(\Omega \setminus \{a\})$ and the singularity of g^2 at $a \in \Omega$ is removable. Prove that then the singularity of g itself is also removable.

5. Suppose h is holomorphic in some neighborhood of $0 \in \mathbb{C}$. Prove that the series

$$\sum_{n=0}^{\infty} \frac{h^{(n)}(z)(-z)^n}{n!}$$

converges in some neighborhood of 0, and its sum is independent of z.

6. Let $Q = \{z \in \mathbb{C} : |\operatorname{Re} z|, |\operatorname{Im} z| < 1\}$, and $\phi : Q \to Q$ be holomorphic. Given that $\phi(0) = 0$, prove that $|\phi'(0)| \leq 1$.

7. Is there an $F \in \mathcal{O}(D_1(0))$ such that for all $\zeta \in \partial D_1(0)$

$$\lim_{z \to \zeta} |F(z)| = \infty\,?$$