MA 53000 QUALIFIER, 8/8/2018

Each problem is worth 5 points. Make sure that you justify your answers. The family of holomorphic functions on an open set $\Omega \subset \mathbb{C}$ is denoted $\mathcal{O}(\Omega)$.

Notes, books, crib sheets, and electronic devices are not allowed.

1. If $\Omega \subset \mathbb{C}$ is open, $f \in \mathcal{O}(\Omega)$, and $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, prove that

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'|^2.$$

2. Compute the following integral (the path of integration is oriented counter-clockwise):

$$\int_{|z|=1/2} \frac{e^z}{z^5 - z^3 + z^2}\, dz.$$

3. Given that $\varphi : [0, \infty) \to \mathbb{C}$ is a bounded continuous function, prove that

$$H(z) = \int_0^\infty \frac{\varphi(t)}{t^2 + z}\, dt$$

defines a function holomorphic on $\mathbb{C} \setminus \{z \in \mathbb{R} : z \le 0\}$.

4. Show that a function $g \in \mathcal{O}(\mathbb{C})$ is $2\pi i$–periodic (i.e., $g(z + 2\pi i) = g(z)$) if and only if there is an $h \in \mathcal{O}(\mathbb{C} \setminus \{0\})$ such that $g(z) = h(e^z)$.

5. Consider a function ϕ holomorphic on $\{z \in \mathbb{C} : |z| > r\}$, where $r \in (0, 1)$. Suppose that there are a real number K and a natural number N such that $|\phi(z)| \le K|z|^N$ for all z, and $|\phi(z)| \le 1$ when $|z| = 1$. Prove that $|\phi(z)| \le |z|^N$ when $|z| \ge 1$.

6. Construct a biholomorphic map between $\{z \in \mathbb{C} : |z - 1| < 1,\ |z - 1/2| > 1/2\}$ and the unit disc. If the map is obtained as a composition of simpler maps, you need not write out explicitly the composition.

7. In this problem $D_r = \{z : |z| < r\}$. Suppose $F, G \in \mathcal{O}(D_1)$, G is injective, $F(0) = G(0)$, and $F(D_1) \subset G(D_1)$. Prove that $F(D_r) \subset G(D_r)$ for all $r < 1$.