MATH 530 Qualifying Exam

January 2017 (S. Bell)

Each problem is worth 20 points

1. Suppose that $f(z)$ is analytic on the complex plane minus a single point z_0. Suppose further that f has a simple pole at z_0 and a removable singularity at infinity. Prove that

$$f(z) = \frac{A}{z - z_0} + B,$$

where A and B are complex constants.

2. Let

$$f(z) = \frac{\log z}{(z^2 + 4)^2},$$

where log denotes a branch of the complex logarithm with branch cut along the negative imaginary axis that agrees with the real logarithm ln on the positive real axis. For a radius $r > 0$, let C_r denote the half circle parametrized by $z(t) = re^{it}$ for $0 \leq t \leq \pi$, and for $a < b$, let $L[a, b]$ denote the line segment on the real line parametrized by $z(t) = t$ for $a \leq t \leq b$.

a) Assume that $r > 0$. Prove that $\int_{C_r} f(z)\, dz$ goes to zero as r goes to infinity and as r goes to zero.

b) Assume that $0 < \epsilon < R$. Note that $\int_{L[\epsilon,R]} f(z)\, dz = \int_\epsilon^R \frac{\ln t}{(t^2+4)^2}\, dt$. Express $\int_{L[-R,-\epsilon]} f(z)\, dz$ in terms of explicit real integrals.

c) Compute the residue of $f(z)$ at $2i$.

d) Finally, use the residue theorem, take limits, and take the real part to compute

$$I = \int_0^\infty \frac{\ln t}{(t^2 + 4)^2}\, dt.$$

3. Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence of distinct points in the unit disc with no limit points in the disc. Prove that the radius of convergence of the power series $\sum_{n=1}^\infty a_n z^n$ is *equal* to one.

4. Prove that the series $\sum_{n=1}^\infty \frac{1}{(z-n)^2}$ converges on the complex plane minus the positive integers to an analytic function with a double pole at each positive integer.

5. Suppose that $f(z)$ is a continuous complex valued function on a disc such that the integral $\int_\gamma f(z)\, dz$ is equal to zero for every contour γ that is the boundary of a square in the disc. Prove that f must be analytic.