**MATH 530 Qualifying Exam**

January 2020 (S. Bell and L. Lempert)

*Each problem is worth 25 points*

**1.** How many zeroes (counted with multiplicity) of the polynomial $10z^{10} + 25z^3 + 13z + 1$ fall outside the unit circle?

**2.** Let $C$ denote the unit circle parametrized in the counterclockwise sense. Compute $\int_C \frac{z}{2z^2+5z+1}\,dz$

**3.** Compute the residue of $\frac{e^{3z}}{1-\cos z}$ at $z = 0$.

**4.** What is the image of the upper half plane under the mapping $L(z) = \frac{z-a}{z-b}$ where $a$ and $b$ are real numbers with $0 < a < b$? Explain.

**5.** In this problem, you will compute the real integral $I = \int_0^\infty \frac{\ln t}{t^3+1}\,dt$ by integrating the complex valued function $f(z) = \frac{\operatorname{Log} z}{z^3+1}$ (where Log denotes the principal branch of the complex log function) around a closed contour $\gamma$ consisting of the directed line segment $L_1$ going from the origin to the point $R > 0$, followed by the circular arc $C_R$ parametrized by $Re^{it}$ as $t$ ranges from zero to $2\pi/3$, then the directed line segment $L_2$ from $Re^{i2\pi/3}$ to the origin.

a) Express the integral of $f(z)$ along the line $L_2$ in terms of real integrals.

b) Show that the integral of $f(z)$ along the circular part $C_R$ of the curve tends to zero as $R \to \infty$.

c) Compute the residue of $f$ at the point $e^{i\pi/3}$.

d) Explain how to use a-c to get a formula for $I$. (Do not simplify.)

**6.** Find a one-to-one conformal mapping of the open unit disc onto the first quadrant of the complex plane.

**7.** Suppose that $u(x, y)$ is harmonic on a disc $\{z : |z| < R\}$ where $R > 1$. Let $M$ denote the maximum of $|u|$ on the unit circle. Prove that there is a real constant $C > 0$ that does not depend on $u$ such that

$$\left|\frac{\partial u}{\partial x}(0,0)\right| \leq CM.$$

**8.** Suppose the $f(z)$ is an entire function such that there exist positive constants $C$ and $R_0 > 1$ such that

$$|f(z)| \leq C \ln |z| \qquad \text{if } |z| > R_0.$$

Prove that $f$ must be constant.