MA 530 Qualifier Exam, August 9, 2021

Each of the seven problems below is worth 5 points. In the problems D stands for the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

In your solutions make sure you justify your claims. Efforts to write neatly will be appreciated. The order of the problems is alphabetical, and is not intended to indicate their levels of difficulty.

1. Compute

$$\int_{-\infty}^{\infty} \frac{e^{3ix} - 3e^{ix} + 2}{x^2}\, dx.$$

2. Construct a biholomorphic map between the strip $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and the first quadrant $\{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z > 0\}$.

3. Expand the function $(z+1)/(z-2)$ in a Taylor series about $z = 0$. Determine the radius of convergence of the series.

4. If ϕ is a holomorphic function on D that vanishes at 0, prove that there is no holomorphic function ψ on $D \setminus \{0\}$ such that $\phi = e^{\psi}$ on $D \setminus \{0\}$.

5. Prove that the complex numbers z for which the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1 + z^{3n}}$$

converges form an open subset of $\mathbb{C}$, and the sum of the series is a holomorphic function there.

6. State and prove Schwarz's lemma. Explain how it can be used to describe a general biholomorphic map $D \to D$ explicitly.

7. Suppose F, G are holomorphic functions on a connected open set $\Omega \subset \mathbb{C}$ and $|F| + |G|$ is constant. Prove that both F and G must be constant.