QUALIFYING EXAM COVER SHEET

August 2022 Qualifying Exams

Instructions: These exams will be "blind-graded" so under the student ID number please use your PUID

ID #: ______________________
(10 digit PUID)

EXAM (circle one) **530** 544 553 554

For grader use:

Points ________ **/ Max Possible**________ **Grade** _________

MATH 530 Qualifying Exam

August 2022 (S. Bell and L. Lempert)

Each problem is worth 20 points

1. Suppose that $\Omega \subset \mathbb{C}$ is a domain and $H(t,s)$ is a twice continuously differentiable homotopy on $[0,1]\times[0,1]$ between two curves in Ω that start at A and end at B. Let γ_s denote the curve parametrized by $z_s(t) = H(t,s)$, $0 \le t \le 1$. Note that all the curves in this family are assumed to start at A and end at B. Hence, $H(0,s) \equiv A$ and $H(1,s) \equiv B$. Let f be an analytic function on Ω and define

$$I(s) = \int_{\gamma_s} f(z)\, dz.$$

Show that $I'(s) \equiv 0$ by completing the following steps. Write out the definition of the path integral in terms of an integral dt. Take the derivative in s under the integral sign

$$I'(s) = \frac{d}{ds}\int_0^1 [\;\cdots] \; dt = \int_0^1 \frac{\partial}{\partial s}[\;\cdots]\; dt,$$

notice that

$$\frac{\partial}{\partial s}[\;\cdots] = \frac{\partial}{\partial t}[\;---\;],$$

and evaluate the integral using elementary calculus. What important fact about f' needs to be known to make this proof work?

2. If $f(z)$ is a continuous complex valued function on a neighborhood of a point a, show that

$$2\pi i f(a) = \lim_{\epsilon \to 0} \int_{C_\epsilon(a)} \frac{f(z)}{z-a}\, dz,$$

where $C_\epsilon(a)$ denotes the counterclockwise circle of radius ϵ about a.

3. Suppose that $\sum_{n=0}^{\infty} a_n z^n$ converges to an entire function $h(z)$, and that each a_n is an integer. Prove that h must be a polynomial.

4. Find all points in the closed unit disc where $z^8 - z^6$ assumes its maximum modulus. Explain.

5. Show that an analytic function $f(z)$ that maps the unit disc into itself that is *not* one-to-one must be such that $|f'(0)| < 1$.

6. Prove that a real valued harmonic function on the whole complex plane that is everywhere positive must be a constant function.

7. Suppose that u is a non-constant real valued harmonic function on the complex plane such that $u(0) = 0$. Prove that there is at least one point on each circle centered at the origin where u vanishes.