MA 530 Qualifier Exam, January 3, 2022

Each of the six problems below is worth 5 points. In the problems D stands for the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

In your solutions make sure you justify your claims. Notes, books, cribsheets, and electronic devices are not allowed. Efforts to write neatly will be appreciated. The order of the problems is alphabetical, and is not intended to indicate their levels of difficulty.

1. Compute

$$\int_L \frac{e^z}{\cos z}\, dz,$$

where L is the lemniscate $|4z^2 - \pi^2| = \pi^2$, oriented as shown above.

2. Construct a biholomorphic map between the domain $\{z \in \mathbb{C} : |z| < 1,\ \mathrm{Re}\, z, \mathrm{Im}\, z > 0\}$ and the half–plane $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$.

3. For what positive numbers R can the function $1/\sin(\pi z)$ be expanded in a Laurent series in the punctured disc $\{z \in \mathbb{C} : 0 < |z - 1| < R\}$? Find the coefficients of $(z-1)^k$ in this expansion for $k \le 2$.

4. If u is a harmonic function in the disc D, and on each circle $\{z \in \mathbb{C} : |z| = r\}$, $0 < r < 1$, it is constant, prove that it is constant on D.

5. If a polynomial is injective on the half plane $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$, show that its degree is 1 or 2.

6. Prove that the sum of the series

$$\sum_{n=1}^{\infty} \frac{3^n}{z^n + z^{-n}}$$

is a holomorphic function on the disc $\{z \in \mathbb{C} : |z| < 1/3\}$.