Qualifying examination

Math 530, August 2024

Prof. A. Eremenko

Each problem is worth 10 points. In your proofs you may refer to any theorem from Ahlfors' book if you state it completely and correctly. Books, notes and electronic devices not permitted.

1. Let f be an entire function which takes real values on the real and imaginary axes. Prove that f is even.

2. Find the residue

$$\mathrm{res}_{z=0}\frac{1}{(e^z-1)^2}.$$

3. Consider the series

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,$$

where $a_0\neq 0$ and $a_n=a_{n-1}-2a_{n-2}$ for $n\geq 2$. Find the radius of convergence of this series.

4. Evaluate the integral

$$\int_{|z|=2}\frac{z^4}{z^5+15z+1}dz,$$

where the circle is parametrized counterclockwise.

5. Consider the polynomial

$$f(z)=z+z^2/2.$$

a) Prove that f is injective in the unit disk $U=\{z:|z|<1\}$.

b) Find the area of the image $f(U)$.

6. Find all solutions of the equation

$$\tan z=2i,$$

and make a picture of them.

7. Let $f = u + iv$ be a non-constant analytic function in some region, where u, v are real valued harmonic functions. Is it possible that

$$u = F \circ v,$$

where F is some continuously differentiable function mapping the real line into itself?

If your answer is positive, give an example, if not, give a proof.