MATH 530 Qualifying Exam

January 2024 (S. Bell)

Each problem is worth 20 points

$D_r(a)$ denotes the open disc of radius r about a

1. Calculate

$$\int_0^\infty \frac{1}{x^n + 1}\, dx$$

for positive integers $n \geq 2$ by integrating a complex function around the closed contour that follows the real axis from the origin to $R > 0$, then follows the circular arc $Re^{i\theta}$ as θ ranges from zero to $2\pi/n$, then returns to the origin via the line segment joining $Re^{2\pi i/n}$ to the origin; let $R \to \infty$. Show all your calculations and explain all limits.

2. Describe the image of the half-strip $\{z = x + iy : -1 < x < 1,\ 0 < y < \infty\}$ under the mapping $f(z) = \dfrac{z-1}{z+1}$.

3. (a) Prove that $f(z) = 1/z$ does not have a complex antiderivative in $\mathbb{C} - \{0\}$.

(b) Find all integers $n = 0, \pm 1, \pm 2, \ldots$ such that the function $g(z) = z^n e^{1/z}$ has a complex antiderivative in $\mathbb{C} - \{0\}$.

4. Let f be an analytic function with a zero of order 2 at z_0. Prove that there exist $\epsilon > 0$ and $\delta > 0$ such that for every w in $D_\epsilon(0) - \{0\}$, the equation $f(z) = w$ has exactly 2 *distinct* roots in the set $z \in D_\delta(z_0) - \{z_0\}$.

5. Prove that there is no analytic function that maps the punctured disc $\{z \in \mathbb{C} : 0 < |z| < 1\}$ one-to-one onto the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.