MATH 530 Qualifying Exam

August 2025, G. Buzzard, S. Bell (Each problem is worth 20 points.)

1. Determine all points $z \in \mathbb{C}$ such that

$$|\cos z|^2 + |\sin z|^2 \leq 1.$$

2. Let $R > 0$ and let f be holomorphic in the disk $D_R = \{z : |z| < R\}$. Define $A : [0, R) \to \mathbb{R}$ as the sup of $\text{Re}(f)$ over the circle of radius r:

$$A(r) = \sup\{\text{Re}(f(z)) : |z| = r\}.$$

Show that if f is nonconstant, then A is strictly increasing on $[0, R)$.

3. Let $a \in (0, 1)$. Evaluate $\displaystyle\int_0^\infty \frac{x^{-a}}{1+x}dx$ as a function of a.

4. Let $\mathcal{F}$ be the set of functions, f, holomorphic in a neighborhood of the unit disk such that

$$\int_0^{2\pi} |f(e^{it})|^2 dt \leq 1.$$

Prove that $\mathcal{F}$ is a normal family in the unit disk: that is, for every sequence $\{f_n\}_{n=1}^\infty$ with each $f_n \in \mathcal{F}$, there is a subsequence $\{f_{n_k}\}_{k=1}^\infty$ that converges uniformly on compact subsets of the unit disk to a function f. The function f is not required to be in $\mathcal{F}$.

Hint: you may use the Cauchy-Schwarz inequality in the form that if g_1, g_2 are continuous functions on the boundary of the unit disk and $\|g_j\| := \left(\int_0^{2\pi} |g_j(e^{it})|^2 dt\right)^{1/2} < \infty$, then $\int_0^{2\pi} |g_1(e^{it})g_2(e^{it})|dt \leq \|g_1\| \, \|g_2\|$.

5. Let $D \subset \mathbb{C}$ be open and connected, let f_n be holomorphic in D for each positive integer n, and suppose f_n converges to f uniformly on D as $n \to \infty$. Suppose also there exists $z_0 \in D$ such that $f(z_0) = 0$ but that f is not identically 0. Prove that there exists n such that f_n has a zero in D.

6. Let f be holomorphic in the punctured unit disk $\mathbb{D} \setminus \{0\}$, let $r \in (0, 1)$, and define $\gamma(t) = re^{it}$ for $t \in [0, 2\pi]$. Define $c = \displaystyle\frac{1}{2\pi i}\int_\gamma f(z)dz$, and let $h(z) = f(z) - \displaystyle\frac{c}{z}$. Prove that h has a holomorphic anti-derivative in $\mathbb{D} \setminus \{0\}$. That is, there exists H holomorphic in $\mathbb{D} \setminus \{0\}$ such that $H' = h$.