MATH 530 Qualifying Exam

January 2025 (S. Bell)

Each problem is worth 25 points

1. Suppose f and g are analytic near a and that g has a double zero at $z = a$ (meaning that $g(a) = 0$, $g'(a) = 0$, and $g''(a) \neq 0$).

a) Show that $g(z) = (z-a)^2 h(z)$ where h has a removable singularity at a. Find the Taylor coefficients for the power series for h about a.

b) Find the principal part of f/g at $z = a$ by computing a small number of coefficients in the Taylor series for f/h. Write a formula for the residue of f/g at $z = a$

2. Suppose that $u(z, s)$ is a continuous real valued function on $\mathbb{C} \times \mathbb{R}$ such that $u(z, s)$ is harmonic in z for each fixed s. Define

$$U(z) = \int_{-1}^{1} u(z, s)\, ds.$$

Prove that U is harmonic on $\mathbb{C}$. (This can be done without taking derivatives.)

3. At what point or points in the closed unit disc does $z^6 + z^4$ attain its maximum modulus? Explain.

4. Suppose Ω is a simply connected domain that is not the whole complex plane and a is a point in Ω. Prove that if f is analytic on Ω and maps Ω into Ω with $f(a) = a$, then $|f'(a)| \leq 1$.

5. a) Find an analytic function $f(z)$ that maps the horizontal strip $\Omega = \{z : 0 < \text{Im } z < 1\}$ one-to-one onto the unit disc.

b) Show that the family of one-to-one conformal mappings of the horizontal strip Ω onto itself is such that, given any two points a_1 and a_2 in the strip, there is a mapping in the family that maps a_1 to a_2.

6. Explain why $\dfrac{\sin z^2}{(z-1)(z+1)}$ has an analytic antiderivative on $\mathbb{C} - [-1, 1]$.

7. Compute

$$\int_\gamma \frac{e^{5z}}{z^7}\, dz,$$

where γ denotes an ellipse with one focus at the origin parameterized in the *clockwise* direction.

8. Suppose $f(z)$ is a complex valued continuous function on the closed unit disc $\{z : |z| \leq 1\}$ that is analytic on $\{z : |z| < 1\}$. Suppose further that f maps the unit circle $\{z : |z| = 1\}$ into the real line. Prove that f must be a real constant function.