QUALIFYING EXAMINATION
AUGUST 1994
MATH 544

1. Assume f_n is a measurable function on $\mathbb{R}$ for $n = 1, 2, \ldots$, and $f_n(x) \to f(x)$, $\forall x$. In each case say whether the additional hypotheses given imply that $\int f_n(x)dx \to \int f(x)dx$ and justify your answer.

(5) a. $\forall n$, $|f_n| \leq 1$ and $m(\{x : f_n(x) \neq 0\}) \leq 1$.

(5) b. $\forall n, \forall x$, $|f_n(x)| \leq \dfrac{1}{1+x^2}$.

(5) c. $\forall n$, $f_n \geq 0$ and $\int f_n(x)dx \leq 1$.

(5) d. $\forall n$, $0 \leq f_n \leq f_{n+1}$ and $\int f_n(x)dx \leq 1$.

(15) 2. Assume f_n is a measurable function on $[0,1]$ for $n = 1, 2, \ldots$, $|f_n| \leq g$, $\forall n$, $\int_0^1 g(x)dx < \infty$, and $F_n(x) = \int_0^x f_n(t)dt$ for x in $[0,1]$. Show that $(F_n)_{n=1}^{\infty}$ has a uniformly convergent subsequence.

3. Let f be a measurable function on a measure space (S, μ), and assume $1 \leq p_1 < p < p_2 < \infty$.

(7) a. Show that if $||f||_{p_1} < \infty$ and $||f||_{p_2} < \infty$, then $||f||_p < \infty$.

(8) b. Show that if $\mu(S) < \infty$ and $||f||_p < \infty$, then $||f||_{p_1} < \infty$.

(10) c. Show that there is a function f on $[0,1]$ with Lebesgue measure such that $||f||_1 < \infty$ and $||f||_p = \infty$, $\forall p > 1$.

4. For f a real-valued function on $[0,1]$ let $f_h(x) = \begin{cases} f(x+h), & x+h \in [0,1] \\ 0, & x+h \notin [0,1] \end{cases}$.

(10) a. Assume $f \in L^p$ and $1 \leq p < \infty$. Show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|h| < \delta \Rightarrow ||f_h - f||_p < \epsilon$.

(7) b. Assume f is continuous and $f(0) = f(1) = 0$. Show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|h| < \delta \Rightarrow ||f_h - f||_\infty < \epsilon$.

(8) c. Prove the converse to b: If $f \in L^\infty$ and if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|h| < \delta \Rightarrow ||f_h - f||_\infty < \epsilon$, then there is a continuous function $\tilde{f}$ such that $\tilde{f}(0) = \tilde{f}(1) = 0$

and $\tilde{f} = f$ almost everywhere. (Hint for c: First show that $\forall \epsilon > 0 \; \exists$ a continuous function g_ϵ such that $||f - g_\epsilon||_\infty < \epsilon$.)

5. Assume f is a real-valued function on $[0, 1]$ and
   (i) f is continuous from the right at each x in $[0, 1)$
   (ii) The left-hand limit, $\lim_{y \to x-} f(y)$, exists for each x in $(0, 1]$.

(5) a. Show that f is bounded.

(10) b. Show that for each $\epsilon > 0$, there is a partition, $0 = x_0 < x_1 < \cdots < x_n = 1$, such that whenever $0 \leq i < n$ and $s, t \in [x_i, x_{i+1})$, then $|f(s) - f(t)| < \epsilon$.

(Note: Hypothesis (ii) cannot be dropped and the conclusion of b would be false if $[x_i, x_{i+1})$ is replaced by $[x_i, x_{i+1}]$.)