QUALIFYING EXAMINATION
MA 544

Spring 1995

Name: ______________________________

Instructions. Standard notation is used throughout. In particular, $\mathbb{R}$ ={reals}, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, I is a compact interval in $\mathbb{R}$, and $|A|$ is the Lebesgue measure of A, a measurable subset of $\mathbb{R}$.

There will be 6 *additional* pages with a problem on each page. Use the space provided for your solution of the problem.

1. Let $\{f_n\} \subset C(I)$ such that for $x \in I$, $f_1(x) \leq f_2(x) \leq \cdots \to f(x)$ pointwise on I. Show that $\{f_n\}$ is equicontinuous on I if and only if $f \in C(I)$.

2. Let $f : \mathbb{R} \to \mathbb{R}_+$ be measurable, and let $0 < r < \infty$. Show that

$$\frac{1}{\frac{1}{|I|}\int_I f} \leq \left(\frac{1}{|I|}\int_I \frac{1}{f^r}\right)^{1/r}$$

for every $I \subset \mathbb{R}$.
(Hint:$|I| = \int_I f^t f^{-t}$.)

3. Let $\{f_n\}$ be a sequence of non-negative measurable functions in $L^p(\mathbb{R})$ for some $1 < p < \infty$. Show that $f_n \to f(L^p)$ if and only if $f_n^p \to f^p(L^1)$.

4. Let $f : \mathbb{R} \to \mathbb{R}_+$ be measurable, and let $\epsilon > 0$. Show that there exists $g : \mathbb{R} \to \mathbb{R}_+$ measurable such that (i) $||f - g||_\infty \leq \epsilon$ and (ii) for every $r \in \mathbb{R}, |\{x : g(x) = r\}| = 0$.

5. Assume that $f \in AC(I)$ for every $I \subset \mathbb{R}$. If both f and f' are in $L^1(\mathbb{R})$ show that (i) $\int_{\mathbb{R}} f' = 0$, and (ii) $f(x) \to 0$ as $|x| \to \infty$.

6. For $f : I \to \mathbb{R}$ let

$$\overline{D}f(x) = \limsup_{h\to 0} \frac{f(x+h) - f(x)}{h},$$
$$\underline{D}f(x) = \liminf_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

If $-K \leq \underline{D}f(x) \leq \overline{D}f(x) \leq K < \infty$ for every $x \in I$, show that $|f(x') - f(x'')| \leq K|x' - x''|$ for every $x', x'' \in I$.