QUALIFYING EXAMINATION

MA 544

SPRING 1996

Name: ______________________________

Instructions. Standard notation is used throughout. In particular, $\mathbb{R}$ ={reals}, $I_0 = [0,1]$, and $C(I_0), BV(I_0), AC(I_0), L^p(I_0)$ are the common function spaces over I_0. For a measurable subset A of $\mathbb{R}$, let $|A|$ denote the Lebesgue measure of A. All functions are assumed to be measurable. If $1 \le p \le \infty$, then p' is the conjugate index, i.e., $1/p + 1/p' = 1$.

There will be 6 *additional* pages with a problem on each page. Use the space provided for your solution of the problem.

1. Let $f \in C(I_0)$. Show that there exists a sequence of polynomials $\{p_n\}$ with integer coefficients such that p_n converges point-wise on I_0 to

$$g(x) = \begin{cases} f(x), & 0 < x < 1 \\ 0, & x = 0, 1. \end{cases}$$

(Hint: You may use without proof the fact that if $f \in C(I_0)$, then such a sequence of polynomials $\{p_n\}$ exists which converges on I_0 uniformly to f iff $f(0)$ and $f(1)$ are integers.)

2. Assume that $f \in AC(I_0)$. Show that $V(x) = V(f; [0, x])$ is also in $AC(I_0)$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $1 \leq p \leq \infty$. Let $\{f_n\} \subset L^{p'}(\mu)$ with $||f_n||_{p'} \leq M < \infty$. Assume that $\{\int_X f_n \phi \, d\mu\}$ converges for every ϕ in a dense subset of $L^p(\mu)$. Show that $\{\int_X f_n \phi \, d\mu\}$ converges for every $\phi \in L^p(\mu)$.

4. Let $f \in L^2(I_0)$, $||f||_2 = 1$ and $\int_0^1 f\,dm \geq \alpha > 0$. If $E_\beta = \{x \in I_0 : f(x) \geq \beta\}$ and $0 < \beta < \alpha$, then $|E_\beta| \geq (\alpha - \beta)^2$.
(Hint: $\alpha \leq \displaystyle\int_{E_\beta} + \int_{I_0 \setminus E_\beta}$.)

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Assume that $||f_n||_p \leq M < \infty$, $n = 1, 2, \ldots$, for some $1 < p < \infty$, and that $f_n \to f$ in measure, i.e., $\mu\{x : |f(x) - f_n(x)| > \delta\} \to 0$ as $n \to \infty$, for every $\delta > 0$. Show that $f_n \to f$ in $L^1(\mu)$.
(Hint: $\phi_n = |f - f_n|$, $E_{\delta,n} = \{x : \phi_n(x) > \delta\}$. Write $\int_X ?\, d\mu = \int_{X \setminus E_{\delta,n}} ?\, d\mu + \int_{E_{\delta,n}} ?\, d\mu$.)

6. Given $(X, \mathcal{M}, \mu)$, $1 \leq p < \infty$, $0 < \eta < p$. If $f_n \to f(L^p)$ and $g_n \to g(L^p)$, show that

$$\lim_{n\to\infty} \int_X |f_n|^{p-\eta} |g_n|^\eta \, d\mu = \int_X |f|^{p-\eta} |g|^\eta \, d\mu.$$