QUALIFYING EXAMINATION
AUGUST 1997
MATH 544

(15 pts) 1. Find each limit and justify your answers.

a. $\lim_{n\to\infty}\int_{\frac{1}{n}}^{n} e^{\frac{1}{n}\sqrt{x}-x}dx.$

b. $\lim_{n\to\infty}\int_{n}^{2n} \frac{e^{\frac{1}{x}}}{x}\,dx.$

c. $\lim_{n\to\infty}\int_{0}^{n} \frac{\cos\frac{x}{n}}{\sqrt{x+\cos\frac{x}{n}}}\,dx.$

(20 pts) 2. Suppose each f_n is an integrable function on a measure space $(S,\mathcal{B},\mu)$ and $\sum_{n=1}^{\infty}\int |f_n|d\mu < \infty.$

a. Show that $\sum_{n=1}^{\infty} f_n(s)$ converges absolutely for almost every s in S.

b. Let $f(s) = \begin{cases} \sum_{n=1}^{\infty} f_n(s), & \text{if the sum converges} \\ 0, & \text{otherwise.} \end{cases}$

Show that f is a measurable function.

c. Show that f is integrable and $\int f d\mu = \sum_{n=1}^{\infty}\int f_n d\mu.$

(15 pts) 3. Let f be integrable on $[0,1]$ and define $F(t) = \int_0^t f(x)dx$. Show that F has bounded variation on $[0,1]$.

(20 pts) 4. Suppose that f_n is a continuous function from $[0,1]$ to itself for $n = 1,2,\ldots, x_n, y_n \in [0,1]$, $f_n(x_n)\to 0$, and $f_n(y_n)\to 1$.

a. Show that $\exists t_n \in [0,1]$ for $n = 1,2,\cdots$ such that $f_n(t_n)\to\frac{1}{2}$.

b. Show that if the family $\{f_n : n = 1,2,\cdots\}$ is equicontinuous, then $\exists t\in[0,1]$ and $n_1 < n_2 < \cdots$ such that $f_{n_k}(t)\to\frac{1}{2}$ as $k\to\infty$.

c. Give an example to show that the statement in b may be false if the equicontinuity hypothesis is omitted.

(15 pts) 5. Suppose f is bounded and measurable on $[0,1]$, $x_0 > 0$, and $\int_0^1 f(t)e^{-xt}dt = 0$, $\forall x > x_0$. Show that $f = 0$ almost everywhere.
Hint: Apply $(\frac{d}{dx})^n$.

(15 pts) 6. Assume each f_n is a uniformly continuous function from $(0,1)$ to $\mathbb{R}$ and $f_n \to f$

uniformly. Show that f is uniformly continuous.