QUALIFYING EXAMINATION
AUGUST 1998

MATH 544 - Profs. Gabrielov/Neugebauer

Name: ______________________________

Instructions. Standard notation is used throughout. In particular, $\mathbb{R}$ ={reals}, $I_0 = [0,1]$, and $C(I_0), L^p(I_0)$ are the common function spaces over I_0. For a measurable subset A of $\mathbb{R}$, let $|A|$ denote the Lebesgue measure of A. All functions are assumed to be measurable.

There will be 6 *additional* pages with a problem on each page. Use the space provided for your solution of the problem.

1. Let $f : I_0 \to \mathbb{R}$. Define for each positive integer n and $x \in I_0$

$$\omega_n(f,x) = \sup\{|f(t_1) - f(t_2| : |x - t_i| \le 1/n, t_i \in I_0, i = 1,2\}.$$

(i) Show that $\lim_{n\to\infty} \omega_n(f,x)$ exists. Call this limit $\omega(f,x)$, the *oscillation* of f at $x \in I_0$.

(ii) Show that, if $\omega(f,x) < \infty$ for every $x \in I_0$, then $\omega(f,x)$ is bounded on I_0.

2. Let $f \in L^1(I_0)$ and let $N = \{x : f(x) = 0\}$. Assume that $|N| > 0$. Show that

$$\left(\int_{I_0} |f|^r\right)^{1/r} \to 0$$

as $r \searrow 0$.
Hint: Apply Hölder's inequality to $\int |f|^r \cdot 1$.

3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Show that $f \in L^1(\mu)$ if and only if

$$\sum_{j=1}^{\infty} \mu\{x : |f(x)| > j\} < \infty.$$

4. Let $f \geq 0$ be in $L^1(I_0)$ and let $I_{k,j} = [(j-1)/2^k, j/2^k)$, $j = 1, \cdots, 2^k$. If

$$f_k(x) = \sum_{j=1}^{2^k} \frac{1}{|I_{k,j}|} \int_{I_{k,j}} f \cdot \chi_{I_{k,j}}(x),$$

show that $f_k \to f$ in L^1.

5. Assume that $\{f_n\}$ is a sequence of absolutely continuous functions on I_0 which satisfies $f_n \to f$ in L^1 and $\{f_n'\}$ is Cauchy in L^1. Show that there exists f^* absolutely continuous on I_0 such that $f^* = f$ a.e.

6. Let F be a closed, nowhere dense subset of I_0 and let

$$h(x) = \alpha \cdot |[0, x] \setminus F|, \quad \alpha = |I_0 \setminus F|^{-1}.$$

Show that
(i) h is a homeomorphism from I_0 onto I_0.

(ii) h is absolutely continuous on I_0.

(iii) h^{-1} is absolutely continuous on I_0 if and only if $|F| = 0$.