# QUALIFYING EXAMINATION

JANUARY 1999

MATH 544 - Profs. Hunt/Gabrielov

**Each problem is worth 10 points.**

**1.** Suppose that $\{f_n\}_{n\geq 1}$ is pointwise bounded and equicontinuous on a compact set $K$.

(a) Show that $\{f_n\}_{n\geq 1}$ is uniformly bounded on $K$.

(b) Show that $f(x) = \inf\{f_n(x) : n \geq 1\}$ is uniformly continuous on $K$.

**2.** Suppose that $f$ is continuous on the interval $0 < x \leq 1$.

(a) Show that there exists a sequence of polynomials that converges pointwise to $f$ on $(0, 1]$.

(b) State a necessary and sufficient condition on $f(x)$ for $x \in (0, 1]$ such that the convergence in (a) may be taken to be uniform on $(0, 1]$. Show that the condition is necessary and sufficient.

**3.** If $f_n$ is measurable for each $n \geq 1$, show that $\limsup_{n\to\infty} f_n$ is measurable.

**4.** (a) Show that $0 \leq a \leq b \leq 2\pi$ implies $\lim_{n\to\infty} \int_a^b \cos nt\, dt = 0.$

(b) Show that $\lim_{n\to\infty} \int_0^{2\pi} f(t) \cos nt\, dt = 0$ for every $f \in \mathrm{L}^1[0, 2\pi]$.

**5.** If $f \in \mathrm{L}^\infty[0, 1]$, show that $\lim_{p\to\infty} \left(\int_0^1 |f|^p dx\right)^{1/p} = \|f\|_\infty.$

**6.** If $f_n \to f$ a.e. on $[0, 1]$ and, given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|E| < \delta$ implies $\int_E |f_n|\, dx < \epsilon$, show that $\lim_{n\to\infty} \int_0^1 |f - f_n|\, dx = 0.$

**7.** (a) Let $g_n(x) = \dfrac{e^{-x/n}}{n}$, $x \geq 0,\ n \geq 1.$

(i) Show that $g_n(x) \in \mathrm{L}^1[0, \infty),\ n \geq 1.$

(ii) Show that the hypothesis of the Lebesgue Dominated Convergence Theorem is not satisfied for $\{g_n\}_{n\geq 1}$.

(b) Let $h_n(x) = n \sin\left(\dfrac{x}{n}\right)$, $0 \leq x \leq \pi,\ n \geq 1.$ Show that the hypothesis of the Lebesgue Dominated Convergence Theorem is satisfied for $\{h_n\}_{n\geq 1}$.

**8.** Use the Vitali Covering Lemma to show that, if $f$ is finite–valued and increasing on $[0, 1], u > 0, \epsilon > 0$, and $m^*$ denotes Lebesgue outer measure, then

$$m^*\left(\left\{x \in [0, 1) : \limsup_{h\to 0^+} \frac{f(x + h) - f(x)}{h} > u\right\}\right) \leq \frac{f(1) - f(0)}{u} + \epsilon.$$