QUALIFYING EXAMINATION
AUGUST 2000
MATH 544–Professor R. Bañuelos

NAME: ____

(PLEASE PRINT CLEARLY)

Instructions: There are a total of 6 problems in this exam with problem 3 containing two parts. A problem appears on each of the following seven (7) pages. Use the space provided for the solutions of the problem.

IMPORTANT: If there is anything in the statements of the problems that is not clear, please ask the person proctoring the exam to clarify it for you.

Problem 1. *(20 pts) Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions with the property that for all $n \geq 1$,*

$$\mu\{x \in X : |f_n(x)| \geq \lambda\} \leq Ce^{-\lambda^2/n}$$

for all $\lambda > 0$. (Here C is a constant independent of n.) Let $n_k = 2^k$. Prove that

$$\limsup_{k \to \infty} \frac{|f_{n_k}|}{\sqrt{n_k log(log(n_k))}} \leq 1, \quad a.e.$$

Problem 2. *(20 pts) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let f_n be a sequence of measurable functions with $f_1 \in L^1(\mu)$ and with the property that*

$$\mu\{x \in X : |f_n(x)| > \lambda\} \leq \mu\{x \in X : |f_1(x)| > \lambda\}$$

for all n and all $\lambda > 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_X \left(\max_{1\leq j\leq n} |f_j| \right) d\mu = 0$$

(Hint: You may assume the fact that $\|f\|_1 = \int_0^\infty \mu\{|f(x)| > \lambda\} d\lambda$.)

Problem 3i. *(10 pts) Let* $(X, \mathcal{F}, \mu)$ *be a finite measure space. Let* $\{f_n\}$ *be a sequence of measurable functions. Prove that* $f_n \to f$ *in measure if and only if every subsequence* $\{f_{n_k}\}$ *contains a further subsequence* $\{f_{n_{k_j}}\}$ *that converges almost everywhere to* f*.*

Problem 3ii. *(10 pts) Let* $(X, \mathcal{F}, \mu)$ *be a finite measure space. Let* $F : \mathbb{R} \to \mathbb{R}$ *be continuous and* $f_n \to f$ *in measure. Prove that* $F(f_n) \to F(f)$ *in measure. (You may assume, of course, that* f_n*,* f*,* $F(f_n)$ *and* $F(f)$ *are all measurable.)*

Problem 4. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space and suppose $f \in L^1(\mu)$ is nonnegative. Suppose $1 < q < \infty$ and let $1 < p < \infty$ be its conjugate exponent ($1/p + 1/q = 1$). Suppose f has the property that*

$$\int_E f d\mu \leq (\mu(E))^{\frac{1}{q}}$$

for all measurable sets E. Prove that $f \in L^r(\mu)$ for any $1 \leq r < p$.

(Hint: Consider $\{x \in X : 2^n \leq f(x) < 2^{n+1}\}$, if you like.)

Problem 5. *(20 pts) Let f be a continuous function on $[-1,1]$. Find*

$$\lim_{n\to\infty} n \int_{-1/n}^{1/n} f(x)\,(1-n|x|)\,dx$$

Problem 6. *(20 pts) Let* $(X, \mathcal{F}, \mu)$ *be a measure space and suppose* $f \in L^p(\mu)$, $1 \leq p < \infty$. *Suppose* E_n *is a sequence of measurable sets satisfying* $\mu(E_n) = \frac{1}{n}$, *for all* n. *Prove that*

$$\lim_{n \to \infty} \left(n^{\frac{p-1}{p}} \int_{E_n} |f| d\mu \right) = 0$$