QUALIFYING EXAMINATION
JANUARY 2001
MATH 544 - Prof. Zink

1. (a) Let $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\pi(x, y) = x, \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}.$$

Show that $\pi(G)$ is open if G is an open subset of $\mathbb{R} \times \mathbb{R}$.

(b) Prove or disprove: If π is the projection mapping defined above, and if F is a closed subset of $\mathbb{R} \times \mathbb{R}$, then $\pi(F)$ is closed.

2. Let both f and g be functions of bounded variation on the closed, bounded interval $[a, b]$. Prove that fg is also of bounded variation on $[a, b]$.

3. Consider the following proposition:

"Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $\mathbb{R}$ which converges at each point of $\mathbb{Q}$, the set of all rational numbers. If $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is equicontinuous, then the sequence converges at every point of $\mathbb{R}$, and the convergence is uniform."

Prove the part that is true, and if there is some portion of the assertion that is false, provide a counterexample.

4. Let g be a real-valued, continuous function on the (finite) interval $[a, b]$.
Definition: If $x \in [a, b]$ and if there exists an element $y \in [a, b]$ such that $x < y$ and $g(x) < g(y)$, then x is a POINT OF ASCENT for g.

Prove the following theorem (due to F. Riesz).

Let $g : [a, b] \to \mathbb{R}$ be continuous, and let A be the set of all points of ascent for g. Then A is an (relatively) open subset of $[a, b]$, and for each of the component (relatively) open intervals of A, (a_k, b_k) (or, perhaps, $[a, b_k)$), $g(a_k) \leq g(b_k)$.

5. Let μ be the Lebesgue measure on the Lebesque measurable subsets of $[0, 1]$, let f be a Lebesgue integrable function, and let

$$\varphi : [1, +\infty) \to \overline{\mathbb{R}} \quad (\text{where } \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\})$$

be given by $\varphi(p) = \|f\|_p$, for all $p \in [1, +\infty)$.

Show that φ is monotonically increasing.

6. Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative, measurable $(\mathcal{S})$ functions on X that converges a.e. to $\mathbf{0}$.
Prove the following:
If there exists a real number M such that

$$\int \sup\{f_1, \cdots, f_n\} d\mu \leq M, \quad \forall n \in \mathbb{N},$$

then

$$\lim_n \int f_n d\mu = 0.$$

7. Let E be a Lebesgue measurable subset of $[0, 1]$, and let μ be the Lebesgue measure. Demonstrate the existence of a point $t \in (0, 1)$ such that

$$\mu(\{x : x \in E \text{ and } x < t\}) = \frac{1}{2}\mu(E).$$