QUALIFYING EXAMINATION

JANUARY 2002

MATH 544 - Prof. Bañuelos

NAME:____

(PLEASE PRINT CLEARLY)

Instructions: There are a total of six (6) problems. A problem appears on each of the following six (6) pages. Use the space provided for the solutions of the problem. Use back pages if you need additional space. If anything is not clear to you in the statements of the problems, please ask the person proctoring the exam. **Each problem is worth ten (10) points.**

Problem 1. *For any two subset A and B of $\mathbb{R}$ (here and below, $\mathbb{R}$ denotes the reals) define $A + B = \{a + b, a \in A, b \in B\}$.*

(i) Suppose A is closed and B is compact. Prove that $A + B$ is closed.

(ii) Give an example that shows that (i) may be false if we only assume that A and B are closed.

Problem 2. *Suppose* $f : [0,1] \to \mathbb{R}$ *is differentiable at every* $x \in [0,1]$ *where by differentiability at* 0 *and* 1 *we mean right and left differentiability, respectively. Prove that* f' *is continuous if and only if* f *is uniformly differentiable. That is, if and only if for all* $\varepsilon > 0$ *there is an* $h_0 > 0$ *such that*

$$\left|\frac{f(x+h) - f(x)}{h} - f'(x)\right| < \varepsilon$$

whenever $0 \leq x, x+h \leq 1,\ 0 < |h| < h_0$.

Problem 3. *Let* $(X, \mathcal{F}, \mu)$ *be a measure space with* $\mu(X) = 1$ *and let* $F_1, \ldots, F_{17}$ *be seventeen (17) measurable subsets of* X *with* $\mu(F_j) = \frac{1}{4}$ *for every* j.

(i) *Prove that five (5) of these subsets must have an intersection of positive measure. That is, if* $E_1, \ldots, E_k$ *denotes the collection of all nonempty intersections of the* $F_j's$ *taken five (5) at a time* ($k \leq 6188$)*, show that at least one of these sets must have positive measure.*

(ii) *Is the conclusion in (i) true if we take sixteen (16) sets instead of seventeen?*

Problem 4. *Let* $f_n : X \to [0,\infty)$ *be a sequence of measurable functions on the measure space* $(X, \mathcal{F}, \mu)$. *Suppose there is a positive constant* M *such that the functions* $g_n(x) = f_n(x)\chi_{\{f_n \le M\}}(x)$ *satisfy* $\|g_n\|_1 \le \frac{A}{n^{4/3}}$ *and for which* $\mu\{x \in X : f_n(x) > M\} \le \frac{B}{n^{5/4}}$, *where* A *and* B *are positive constant independent of* n. *Prove that*

$$\sum_{n=1}^{\infty} f_n < \infty, \quad a.e.$$

Problem 5. *Let $\{g_n\}$ be a bounded sequence of functions on $[0,1]$ which is uniformly Lipschitz. That is, there is a constant M (independent of n) such that for all n, $|g_n(x) - g_n(y)| \le M|x-y|$ for all $x, y \in [0,1]$ and $|g_n(x)| \le M$ for all $x \in [0,1]$.*

(i) Prove that for any $0 \le a \le b \le 1$,

$$\lim_{n\to\infty} \int_a^b g_n(x) \sin(2n\pi x)dx = 0.$$

(ii) Prove that for any $f \in L^1[0,1]$,

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x) \sin(2n\pi x)dx = 0.$$

Problem 6. *Let* $\{f_n\}$ *be a sequence of nonnegative functions in* $L^1[0,1]$ *with the property that* $\int_0^1 f_n(t)dt = 1$ *and* $\int_{1/n}^1 f_n(t)dt \leq 1/n$ *for all* n. *Define* $h(x) = \sup_n f_n(x)$. *Prove that* $h \notin L^1[0,1]$.