MATH 544 QUALIFYING EXAM JANUARY 2004 PROF. SA BARRETO

STUDENT NAME

STUDENT NUMBER

CAMPUS

INSTRUCTIONS

1) This exam has seven questions worth a total 100 points. Each question is stated on a separate sheet of paper and there are two more pages on which to write your solution. Please, answer each problem on the corresponding sheets. You should not need more paper, but if you do, please make sure each sheet has you name on it and that it is stapled next to the problem for which it was used.

2) The number in parenthesis at the beginning of a question, or an item in a question, indicates the value of that particular problem.

3) Justify your solutions. Answers without justifications will have ZERO value.

1) Let $Y = C([0,1])$ equipped with the distance

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

a)(5) Prove that Y is not a complete metric space.

b)(5) Let $\widetilde{Y}$ be the completion of Y; that is $\widetilde{Y}$ is the family of equivalence classes of Cauchy sequences in Y, where two Cauchy sequences $\{z_n\}$ and $\{w_m\}$ are in Y are equivalent if $d(z_n, w_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Which space is $\widetilde{Y}$? Justify your answer.

2) Let $f_n \in C^1([0,1])$, $n = 1, 2, ...$, let f_n' denote the derivative of f_n, and assume that

i) $||f_n'||_{L^p([0,1])} \leq 1, \quad n = 1, 2, ...$ for some $p > 1$ independent of n

ii) $|f_n(0)| \leq 1, \quad n = 1, 2, ...$

a)(10) Prove that f_n has a uniformly convergent subsequence.

b)(10) Is this true if we replace (i) with $||f_n'||_{L^1([0,1])} \leq 1, \quad n = 1, 2, ...$?

3) Let $f \in L^1(\mathbb{R})$ satisfy

$$(I) \qquad f(x) = 0 \quad \text{if} \quad |x| > 1,$$

$$(II) \int_{\mathbb{R}} f(x)\, x^k \;\; dx = 0, \quad k = 0, 1, 2, ...$$

a)(10) Prove that $f = 0$ a.e.

b)(10) Does your argument apply if (I) is replaced with the milder condition

$$(I*) \qquad |x^k f| \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{for every} \quad k \in \mathbb{N}?$$

Justify your answer.

4)(10) Let $\epsilon \in (0,1)$. Construct a closed subset $S_\epsilon \subset [0,1]$ which has empty interior but has Lebesgue measure greater than ϵ.

5)(10) Let μ be a positive measure on X and let $f : X \longrightarrow [0,\infty]$ be measurable with $f \in L^1(X,\mu)$. Compute

$$\lim_{n\to\infty} \int_X n \arctan\left[\left(\frac{f}{n}\right)^{\alpha}\right] \, d\mu, \quad \alpha \in (0,\infty).$$

Make sure you explain your computations.

6) Let $f : [0,1] \longrightarrow [0,1]$ be Lebesgue measurable.

a)(5) Construct a sequence of simple functions s_n, $n = 1,2,3,...$ such that

i) $0 \leq s_1 \leq s_2 \leq ... \leq f$

ii) $s_{n+1} - s_n(x) \leq 2^{-n-1}$

ii) $f(x) - s_n(x) \leq 2^{-n}$ for $n = 1,2,...$

b)(5) Let

$$A_n(f) = \{(x,y) \in [0,1] \times [0,1] : 0 < y < s_n(x)\} \quad \text{and}$$
$$\Gamma_n(f) = \{(x,y) : s_n(x) \leq y \leq s_n(x) + 2^{-n}\}.$$

Show that $A_n(f)$ and $\Gamma_n(f)$ are Lebesgue measurable subsets of $[0,1] \times [0,1]$.

c)(5) Compute the measure of $A(f) = \bigcup_{n=1}^{\infty} A_n(f)$ and interpret your result geometrically.

d)(5) Compute the measure of $\Gamma(f) = \bigcap_{n=1}^{\infty} \Gamma_n(f)$ and interpret your result geometrically.

7)(a)(5) Let $(X,\mathcal{M})$ is a measure space and μ is a positive measure on $\mathcal{M}$. Let f be a complex valued measurable function. Show that

$$||f||_{L^\infty(\mu)} \leq \liminf_{p\to\infty} ||f||_{L^p(\mu)}$$

(b)(5) If there exists $r > 1$ such that $||f||_{L^r(\mu)} < \infty$ show that

$$||f||_{L^\infty(\mu)} = \lim_{p\to\infty} ||f||_{L^p(\mu)}.$$