QUALIFYING EXAMINATION
AUGUST 2005
MATH 544–R. Bañuelos

Student ID: ____

(PLEASE PRINT CLEARLY)

Instructions: There are a total of 7 problems in this exam. A problem appears on each of the following eight (8) pages. Problems 1-6 are each worth **20 points** and Problem 7 is worth 10 points for a total possible of 150 points. Partial credit, when applicable, will be given **only** in increments of **5 pints.** Use the space provided for the solutions, using back pages as needed.

Problem 1.

(i) **(5–pts)** Define, carefully, what it means for a function $f : [0, 1] \to \mathbb{R}$ to be of bounded variation.

(ii) **(5–pts)** Define, carefully, what it means for a function $f : [0, 1] \to \mathbb{R}$ to be absolutely continuous.

(iii) **(5–pts)** Suppose f is of bounded variation on $[0, 1]$. Prove that so is e^f.

(iv) **(5–pts)** Suppose f is absolutely continuous on $[0, 1]$. Prove that so is e^f.

Problem 2. **(20–pts)** Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and in $L^1(\mathbb{R})$. Suppose that

$$\int_a^b f(x)dm(x) \geq 0, \quad \text{for all} \;\; a, b \in \mathbb{R}, a \leq b.$$

Prove that $f \geq 0$ a.e.

Problem 3. (20–pts) Prove that the following limit exists

$$\lim_{n\to\infty} \int_0^\infty \frac{e^{-x}\cos x}{nx^2 + \frac{1}{n}}\, dx,$$

and find it, justifying all your steps.

Problem 4. **(20–pts)** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions on it satisfying:

(i)

$$\int_X |f_k|^2 d\,\mu \leq M, \qquad \text{for all k}$$

(ii)

$$\int_X f_j f_k d\mu = 0, \quad \text{for all } j \neq k,$$

where M is a finite constant independent of n. For each $n = 1, 2, \ldots$, set $S_n = \sum_{k=1}^{n} f_k$, Prove that

$$\lim_{n\to\infty} \frac{S_{n^2}}{n^\alpha} = 0, \quad a.e.$$

for all $\alpha > 3/2$. (Careful, careful! By S_{n^2} we mean $\sum_{k=1}^{n^2} f_k = f_1 + f_2 + \cdots + f_{n^2}$.)

Problem 5. **(20–pts)** Let $f : [0,1] \to \mathbb{R}$ be Lebesgue measurable with $f > 0$, a.e. Let $\{E_n\}$ be a sequence of measurable sets in $[0,1]$ with the property that

$$\lim_{n\to\infty} \int_{E_n} f(x)dx = 0. \tag{1}$$

Prove that $\lim_{n\to\infty} m(E_n) = 0$.

Problem 6. **(20–pts)** Let f be Lebesgue measurable on $[0,1]$ with the property that $\|f\|_2 = 1$ and $\|f\|_1 = \frac{1}{2}$. Prove that

$$\frac{1}{4}(1-\lambda)^2 \leq m\{x \in [0,1] : |f(x)| \geq \frac{\lambda}{2}\},$$

for all $0 \leq \lambda \leq 1$. Here, m denotes the Lebesgue measure on $[0,1]$.
Hint: Split the integral of $|f|$ into two pieces.

Problem 7. **(10–pts)** Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Fix $1 \leq n \leq m$ and let $E_1, \ldots, E_m$ be measurable sets with the property that almost every $x \in X$ belongs to at least n of these sets. Prove that at least one of these sets must have μ measure greater than or equal to n/m.