QUALIFYING EXAMINATION
JANUARY 2005

MATH 544 - Prof. Neugebauer

Name: ______________________________

Instructions. Standard notation is used throughout. There will be 5 problems with the space provided for your solution. I have also included a 6^{th} problem that is only intended to test your mathematical "intuition" and "feeling". It will have no influence upon the outcome of this examination. You may even leave it blank.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p < \infty$. If $\{f_n, f\} \in L^p(\mu)$ and $\int_X f_n g d\mu \to \int_X f g d\mu$ for every $g \in L^{p'}(\mu), 1/p + 1/p' = 1$, show that

$$||f||_p \leq \liminf ||f_n||_p.$$

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, $f : X \to \mathbb{R}$ $\mathcal{M}$- measurable, and let $1 \leq p_1 < p_2 < \infty$. Assume there exist constants $0 < c_1, c_2 < \infty$ such that

$$\mu\{x : |f(x)| > y\} \leq \frac{c_j}{y^{p_j}}, j = 1, 2,$$

for every $y > 0$. Show that $f \in L^p(\mu), p_1 < p < p_2$.
(Hint: $||f||^p = p \int_0^\infty y^{p-1} \mu\{x : |f(x)| > y\} dy$.)

3. Let $f_n : I \to \mathbb{R}_+$ be non-decreasing on $I = [a, b]$ with $||f_n||_\infty \leq M < \infty, n = 1, 2, \cdots$. Assume that $\{f_n\}$ converges on a dense subset of I. Show that $\{f_n\}$ converges at every point of I except perhaps a countable set.

4. Let $\{f_n\} \subset AC(I), I = [a, b]$. Assume that $f_n \to f(L^1)$ and $\{f_n'\}$ is Cauchy (L^1). Show that there exists $g \in AC(I)$ such that $f(x) = g(x), a.e. x \in I$.

5. Let $f : I \to \mathbb{R}, I = [a, b]$, and let $M \in \mathbb{N}$. Show that the following two statements are equivalent.
   1. $f \in AC(I)$.
   2. For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for every collection of intervals $\{J_i = [a_i, b_i] \subset I\}$ with $\sum \chi_{J_i}(x) \leq M$ and $\sum |J_i| \leq \delta$, we have $\sum |f(b_i) - f(a_i)| \leq \epsilon$.

6. What can be said in problem 5 if $M = \infty$ is allowed?

REMARK: You will only be graded on the first 5 problems even if you have no comments on problem 6.