QUALIFYING EXAMINATION
January 2007
MATH 544–R. Bañuelos

PUID: ____________________

(PLEASE PRINT CLEARLY)

Instructions: This exam consist of 6 problems. A problem appears on each of the following six (6) pages. Some problems have multiple parts which may or may not be related to each other. Each problem is worth 20 points for a maximum total of 120 points. Use the space provided for the solutions of the problem.

IMPORTANT: If there is anything in the statements of the problems that is not clear, please ask the person proctoring the exam to clarify it for you.

Problem 1. (20 points) Let $f : [0,1] \to \mathbb{R}$.

(i) (5 points) Define what it means for f to be absolutely continuous.

(ii) (5 points) Define what it means for f to be of bounded variation.

(iii) (10 points) Let $V_f(0,x)$ be the total variation of f on $[0,x]$. Prove that if f is absolutely continuous on $[0,1]$ so is $V_f(0,x)$.

Problem 2. (20 points)

(i) (10 points) Suppose that $f : [0, 1] \to \mathbb{R}$ is non-decreasing with $f(0) = 0$ and $f(1) = 1$. For $a > 0$, let A be the set of all $x \in (0, 1)$ for which

$$\limsup_{h \to 0} \frac{f(x+h) - f(x)}{h} > a.$$

Prove that $m^*(A) < \frac{1}{a}$, where m^* denotes the Lebesgue outer measure.

(ii) (10 points) Prove that there is no Lebesgue measurable set A in $[0, 1]$ with the property that $m\,(A \cap I) = \frac{1}{4} m(I)$ for every interval I.

Hint: Consider the function $f(x) = \chi_A(x)$

Problem 3. (20 points) Let $\{E_j\}_{j=1}^{\infty}$ be Lebesgue measurable sets in $[0,1]$ and let $E = \cup_{j=1}^{\infty} E_j$ and suppose there is an $\varepsilon > 0$ such that $\sum_{j=1}^{\infty} m(E_j) \leq m(E) + \varepsilon$.

(i) (10 points) Show that for all measurable sets $A \subset [0,1]$,

$$\sum_{j=1}^{\infty} m(A \cap E_j) \leq m(A \cap E) + \varepsilon$$

(ii) (10 points) Let A be the set of all $x \in [0,1]$ which are in a least two of the $E_j's$. Prove that $m(A) \leq \varepsilon$.

Problem 4. (20 points) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n : X \to [0, \infty)$ be a sequence measurable functions and suppose that $\|f_n\|_p \leq 1$, $1 < p < \infty$, and that $f_n \to f$ a.e. Prove:

(i) (5 points) $f \in L^p(\mu)$

(ii) (15 points) $\|f_n - f\|_1 \to 0$, as $n \to \infty$.

Problem 5. (20 points) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{g_n\}$ be a sequence of non–negative measurable function with the property that $g_n \in L^1(\mu)$ for every n and $g_n \to g$ in $L^1(\mu)$. Let $\{f_n\}$ be another sequence of non–negative measurable functions on $(X, \mathcal{F}, \mu)$.

(i) (10 points) If $f_n \leq g_n$ a.e. for every n, prove that

$$\limsup_{n\to\infty} \int_X f_n \, d\mu \leq \int_X \limsup_{n\to\infty} f_n \, d\mu.$$

Hint: Start by considering a subsequence $f_{n'}$ such that

$$\lim_{n'\to\infty} \int_X f_{n'} \, d\mu = \limsup_{n\to\infty} \int_X f_n \, d\mu$$

and let $g_{n''}$ be a subsequence of $g_{n'}$ such that $g_{n''} \longrightarrow g$ a.e.

(ii) (10 points) If $f_n \to f$ a.e. and if $f_n \leq g_n$ a.e. for all n, then $||f_n - f||_1 \to 0$ as $n \to \infty$.

Problem 6. (20 points) Let $f \in L^1(\mathbb{R})$. Consider the function

$$F(x) = \int_{\mathbb{R}} e^{ix\,t} f(t)\; dt.$$

(i) (10 points) Show that $F \in L^\infty(\mathbb{R})$ and that F is continuous at every $x \in \mathbb{R}$. Moreover, if $|t|^k f(t) \in L^\infty(\mathbb{R})$ for all $k \geq 1$, show that F is infinitely differentiable, i.e., $F \in C^\infty(\mathbb{R})$.

(ii) (10 points) Suppose f is continuous as well as in $L^1(\mathbb{R})$. Show that $\lim_{|x|\to\infty} F(x) = 0$.

Hint: Using $e^{-i\pi} = -1$, write $F(x) = \frac{1}{2}\int_{\mathbb{R}} \left(e^{ixt} - e^{ixt-i\pi}\right) f(x) d\mu$