Math 544 Qualifying Exam, January 2009, L. Lempert

Each problem is worth 5 points.

1. Let $a < b$ be real numbers, $g_i : \mathbb{R} \to [a, b]$ arbitrary functions, $i \in \mathbb{N}$, and $h : \mathbb{R} \to \mathbb{R}$ continuous. Supposing that the g_i converge uniformly, prove that $h \circ g_i$ also converge uniformly.

   Does the same hold if instead of $g_i : \mathbb{R} \to [a, b]$ we require only $g_i : \mathbb{R} \to \mathbb{R}$?

2. Suppose $f \in L^1(\Omega, \mathcal{A}, \mu)$ and $f(x) \neq 0$ for almost every $x \in \Omega$. Prove that μ is σ–finite.

3. Let f be an everywhere finite measurable function on a measure space $(\Omega, \mathcal{A}, \mu)$, such that for every continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ the composition $\alpha \circ f$ is integrable. Prove that ess sup $|f| < \infty$.

4. Let $a < b$ be real numbers and $\phi_n : [a, b] \to \mathbb{R}$ a sequence of increasing, absolutely continuous functions. Show that if the series $\sum_{n=1}^{\infty} \phi_n$ converges pointwise, then its sum is also absolutely continuous.

5. Fix $p \in [1, \infty)$, and for $j = 1, 2, \ldots$ define functions $\omega_j : l^p \to \mathbb{R}$ by

   $$\omega_j(x_1, x_2, \ldots) = \sum_{i=j}^{\infty} |x_i|^p.$$

   Let $C \subset l^p$ be a closed, bounded set, such that the functions $\omega_j|C$ converge uniformly. Prove that any sequence in C contains a convergent subsequence.

6. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, $p_0 \in [1, \infty)$, and $\psi \in L^{p_0}(\Omega, \mathcal{A}, \mu)$. Prove that the function

   $$[1, p_0] \ni p \mapsto ||\psi||_p$$

   is continuous.

7. Suppose $X_1 \subset X_2 \subset \ldots \subset \mathbb{R}$ is an increasing sequence of subsets and $X = \bigcup_{k=1}^{\infty} X_k$. Denoting outer Lebesgue measure by m^*, prove

   $$\lim_{k \to \infty} m^*(X_k) = m^*(X).$$