Name ____________________

MATH 544 Qualifying Examination
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There are 6 numbered problems, each of which is worth 20 points. All answers must be justified. Use the space provided or the backs of the pages. Please ask if you do not understand some of the terminology or notation. Also please ask (discreetly) if you are not sure whether you are allowed to use a particular theorem. If you are asked to prove a result which follows trivially from a theorem you know, you may not use that theorem.

1. a. Let $f\colon A \to \mathbb{R}$, be continuous, where A is a closed bounded subset of $\mathbb{R}$. Show that f is uniformly continuous.

   b. Give an example of a closed subset A of $\mathbb{R}$ and a continuous $f\colon A \to \mathbb{R}$ such that f is not uniformly continuous.

c. Give an example of a bounded subset A of $\mathbb{R}$ and a continuous $f\colon A \to \mathbb{R}$ such that f is not uniformly continuous.

2. Let $f_n(x) = \left(\dfrac{x}{x+1}\right)^n \sin x$ for $x > 0$.

a. Does $\langle f_n \rangle$ converge uniformly on $(0, \infty)$?

b. Is $\{f_n\colon n = 1, 2, \dots\}$ (pointwise) equicontinuous on $(0, \infty)$?

c. Does $\langle f_n \rangle$ converge uniformly on $(0,1)$?

d. Is $\{f_n : n = 1, 2, \ldots\}$ (pointwise) equicontinuous on $(0,1)$?

3. a. Show that every absolutely continuous function on $[0,1]$ is of bounded variation.

b. Show that if f and g are absolutely continuous functions on $[0,1]$, then fg is also absolutely continuous.

4. Let $E_n, n = 1, 2, \ldots,$ be measurable subsets of a measure space (X, μ) such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Let $A = \{x \in X$: There are infinitely many values of n with $x \in E_n\}$.

   a. Show that A is measurable.

b. Show that $\mu(A) = 0$.

5. Find the indicated limits (some of which may be infinite).

a. $$\lim_{n\to\infty} \int_0^\infty \frac{\tan^2 x}{(n+\tan^2 x)(x^2+1)}\, dx.$$

b. $\displaystyle \lim_{n\to\infty} \int_n^\infty \frac{\cos(x+n)}{x^2+1}\,dx.$

c. $\displaystyle \lim_{n\to\infty} \frac{1}{n} \int_n^\infty \cos^2\left(\frac{x}{n^2}\right)\, e^{-\frac{x^2}{n^3}}\,dx$

6. Let $\mathcal{P} = \left\{ \sum_{m=0}^{N} a_m \cos mx + \sum_{n=1}^{N} b_n \sin nx \colon N = 1, 2, \ldots, a_m, b_n \in \mathbb{R} \right\}$, a set of functions on $[0, 2\pi]$.

   a. Show that $\mathcal{P}$ is uniformly dense in

$$\{f \in C([0, 2\pi]) \colon f(0) = f(2\pi)\},$$

   where $C([0, 2\pi])$ is the set of continuous $\mathbb{R}$–valued functions on $[0, 2\pi]$.

b. Show that if f is a bounded measurable function on $[0, 2\pi]$, and if

$$\int_0^{2\pi} f(x) \cos mx dx = 0 = \int_0^{2\pi} f(x) \sin nx dx,$$

$m = 0, 1, 2, \ldots, n = 1, 2, \ldots$, then $f = 0$ almost everywhere.