MA 544 QUALIFYING EXAMINATION
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Antônio Sá Barreto

Identifier:
(Please print clearly in case the front cover gets lost)

PLEASE FOLLOW THESE INSTRUCTIONS:

1) This exam booklet contains 7 problems and 15 pages. The value of each question is indicated next to its statement. Try to write all the solutions on this booklet. If you need extra paper, it will be provided to you. Do not use your own paper. Use one sheet of extra paper per problem and clearly indicate which question you used it for. Staple the extra sheets to this booklet.

2) No questions are allowed during the exam. If you believe there is something wrong with a particular problem, indicate what it is and/or give a counterexample.

3) Thoroughly justify every step of your answers.

4) The result of unsolved questions may be used in the solution of another one, without penalty.

5) No notes or books may be consulted.

1)(20 points) Prove that every open subset $\mathcal{O} \subset \mathbb{R}$ contains a subset which is not Lebesgue measurable.

Continuation of Problem 1

2)(30 points) Let $f : [0,1] \longrightarrow \mathbb{R}$ be an absolutely continuous function. Prove that if $A \subset [0,1]$ is Lebesgue measurable so is $f(A)$.

Continuation of Problem 2

3) Let $(X, \mathcal{S}, \mu)$ be a measure space and let $p \in (1, \infty)$. Let $f_n \in L^p(X, \mu)$, $n \in \mathbb{N}$, be such that $||f_n||_p \leq 1$, and suppose that $f_n \longrightarrow f$ a.e. Use the steps below to prove that

$$(1) \qquad \lim_{n\to\infty} \int_X f_n g \, d\mu = \int_X fg \, d\mu, \text{ for all } g \in L^q(X,\mu), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

a)(10 points) Show that $f \in L^p(X, \mu)$ and $||f||_p \leq 1$.

b) (10 points) Show that if f_n is real valued, then for all real valued $g \in L^q(X, \mu)$ with $1/p + 1/q = 1$, and $\epsilon > 0$, one has

$$f_n g \leq \frac{\epsilon^p |f_n|^p}{p} + \frac{|g|^q}{q\epsilon^q}.$$

c) (10 points) Show that

$$\frac{\epsilon^p}{p}||f||_p^p + \frac{1}{q\epsilon^q}||g||_q^q - \int_X fg \, d\mu \leq \frac{\epsilon^p}{p} + \frac{1}{q\epsilon^q}||g||_q^q - \limsup_{n\to\infty} \int_X f_n g \, d\mu.$$

and hence

$$\limsup_{n\to\infty} \int_X f_n g \, d\mu \leq \int_X fg \, d\mu.$$

d)(10 points) Prove equation (1) above.

Continuation of Problem 3

4) Let $(X, \mathcal{S}, \mu)$ be a measure space.

a)(15 points) Let $\{E_n \in \mathcal{S}, n \in \mathbb{N}\}$ and let χ_{E_n} denote the characteristic function of E_n. Show that $\limsup_{n\to\infty} \chi_{E_n} = \chi_E$, where $E \in \mathcal{S}$, and find an explicit formula for E.

b)(15 points) Let μ and ν be measures on $\mathcal{S}$ and assume that $\nu(X) < \infty$. Suppose that for all $E \in \mathcal{S}$ such that $\mu(E) = 0$, then one also has $\nu(E) = 0$. Prove that for any $\epsilon > 0$ there exists $\delta > 0$ such that if $\mu(E) < \delta$ then $\nu(E) < \epsilon$.

c)(10 points) Prove that if $f \in L^p(X, \mu)$, then

$$\lim_{\lambda\to\infty} \lambda^p \mu(\{x : |f(x)| > \lambda\}) = 0.$$

Continuation of Problem 4

5)(20 points) Let $(X, \mathcal{S}, \mu)$ be a measure space and let $p \in (1, \infty)$. If $\mu(X) < \infty$ and there exist $\lambda_0 > 0$, $C > 0$ and $\epsilon > 0$ such that for all $\lambda \geq \lambda_0$,

$$\lambda^p (\log \lambda)^{1+\epsilon} \mu(\{x : |f(x)| > \lambda\}) \leq C,$$

prove that $f \in L^p(X, \mu)$.

Continuation of Problem 5

6) (20 points) Let $\alpha > 1$. Compute the limit

$$\lim_{n\to\infty} \int_0^n (1 + n^{-1}x)^n e^{-\alpha x}\, dx$$

Continuation of Problem 6

7) For any Lebesgue set $\Gamma \subset (0,\infty)$ let

$$\mu(\Gamma) = \int_\Gamma \frac{1}{x}\, dx.$$

a) (10 points) Let $\mathcal{L}^+$ denote the Lebesgue sets contained in $(0,\infty)$, then μ is a measure on $((0,\infty),\mathcal{L}^+)$, and for any measurable $f : (0,\infty) \longrightarrow [0,\infty]$,

$$\int_{(0,\infty)} f(\alpha x)\, d\mu = \int_{(0,\infty)} f(x)\, d\mu, \quad \alpha \in (0,\infty).$$

b)(20 points) For $f \in L^p((0,\infty), d\mu)$, $p \in [1,\infty)$, and $g \in L^1((0,\infty), d\mu)$, let

$$f \bullet g(x) = \int_{(0,\infty)} f\left(\frac{x}{y}\right) g(y)\, d\mu.$$

Show that $||f \bullet g||_p \leq ||g||_1.||f||_p$.

Continuation of Problem 7