**MA 54400 - Qualifying Exam**

**August 8, 2011**

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| Problem | Score | Max. pts. |
|---|---|---|
| **1** | | 25 |
| **2** | | 20 |
| **3** | | 30 |
| **4** | | 25 |
| **Total** | | 100 |

1. Let $f \in L^1(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x) \cos(tx)\ dx$.
   (a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
   (b) Prove the following *Riemann-Lebesgue Lemma*:

$$\lim_{t \to \infty} F(t) = 0.$$

   *(Hint: Start by proving the statement for $f = \chi_{[a,b]}$.)*

2. (a) Suppose that $f_k, f \in L^2(E)$, with $E$ a measurable set, and that

$$\int_E f_k g \to \int_E f g \quad \text{as } k \to \infty \tag{1}$$

for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \to \|f\|_2$, show that $f_k$ converges to $f$ in $L^2$, i.e.

$$\int_E |f_k - f|^2 \to 0 \quad \text{as } k \to \infty.$$

(b) Provide an example of a sequence $f_k \in$ Ł$^2$ and a function $f \in$ Ł$^2$ satisfying (1), but such that $f_k$ does NOT converge to $f$ in $L^2$.

3. A bounded function $f$ is said to be of bounded variation on $\mathbb{R}$ if it is of bounded variation on any finite sub-interval $[a, b]$, and moreover $A \stackrel{def}{=} \sup_{a,b} V[a, b; f] < \infty$. Here, $V[a, b; f]$ denotes the total variation of $f$ over the interval $[a, b]$. Show that:

   (a) $\int_{\mathbb{R}} |f(x+h) - f(x)|\, dx \leq A|h|$ for all $h \in \mathbb{R}$.

   *Hint: For* $h > 0$ *write*
   $$\int_{\mathbb{R}} |f(x+h) - f(x)|\, dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)|\, dx.$$

   (b) $\left| \int_{\mathbb{R}} f(x)\varphi'(x)\, dx \right| \leq A$, where $\varphi$ is any function of class $C^1$, of bounded variation, compactly supported, with $\sup_{x\in\mathbb{R}} |\varphi(x)| \leq 1$.

4. (a) Prove the following *Generalized Hölder Inequality*: Assume $1 \le p_j \le \infty$, $j = 1, \dots, n$, with $\sum_{j=1}^n 1/p_j = 1/r \le 1$. If $E$ is a measurable set and $f_j \in L^{p_j}(E)$ for $j = 1, \dots, n$, then $\prod_{j=1}^n f_j \in L^r(E)$ and

$$\left\| \prod_{j=1}^{n} f_j \right\|_r \le \prod_{j=1}^{n} \|f_j\|_{p_j}.$$

(b) Use part (a) to show that if $1 \le p, q, r \le \infty$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, $f \in L^p(\mathbb{R})$, and $g \in L^q(\mathbb{R})$, then

$$|(f * g)(x)|^r \le \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q \, dy.$$

(Recall that $(f * g)(x) = \int f(y) g(x-y) \, dy$.)

(c) Prove *Young Convolution Theorem:* Assume that $p, q, r, f, g$ are as in part (b). Then $f * g \in Ł^r(\mathbb{R})$ and

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$