MA 54400 - Qualifying Exam

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Prof. Donatella Danielli

Problem	Score	Max. pts.
1		20
2		25
3		30
4		25
Total		100

In order to receive full credit, you need to show your work and justify your arguments.

1. Let $f(x, y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $\frac{\partial f}{\partial x}(x, y)$ is a bounded function of (x, y). Prove that $\frac{\partial f}{\partial x}(x, y)$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x, y)\,dy = \int_0^1 \frac{\partial f}{\partial x}(x, y)\,dy.$$

2. Let f be of bounded variation on $[a,b]$, $-\infty < a < b < \infty$. If $f = g+h$, with g absolutely continuous and h singular, show that

$$\int_a^b \phi \, df = \int_a^b \phi f' \, dx \; + \; \int_a^b \phi \, dh$$

for all functions ϕ continuous on $[a,b]$.
Hint: A function h is said to be singular if $h' = 0$ a.e.

3. Let $E \subset \mathbb{R}$ be a measurable set, and let K be a measurable function on $E \times E$. Assume there exists a positive constant C such that

(1) $$\int_E K(x,y)\, dx \le C, \qquad \text{a.e. } y \in E,$$

and

(2) $$\int_E K(x,y)\, dy \le C, \qquad \text{a.e. } x \in E.$$

Let $1 < p < \infty,\ f \in L^p(E)$, and define

$$Tf(x) = \int_E K(x,y) f(y)\, dy.$$

(a) Prove that $Tf \in L^p(E)$ and

(3) $$\|Tf\|_p \le C\|f\|_p.$$

(b) Is (3) still valid if $p = 1$ or $p = \infty$? If so, are assumptions (1) and/or (2) needed?

4. Let f be a nonnegative measurable function on $[0,1]$ satisfying

$$(\star) \qquad |\{x \in [0,1] \mid f(x) > \alpha\}| < \frac{1}{1+\alpha^2}, \qquad \alpha > 0.$$

(a) Determine the values of $p \in [1, \infty)$ for which $f \in L^p([0,1])$.

(b) If p_0 is the minimum value of p for which f may fail to be in L^p, give an example of a function f which satisfies $(\star)$, but which is not in L^{p_0}.