QUALIFYING EXAMINATION
January 2013
MATH 544–R. Bañuelos

Student ID: ______________________________

(PLEASE PRINT CLEARLY)

Instructions: There are a total of 6 problems in this exam. A problem appears on each of the following pages. Problems are worth **20 points** each. Use the space provided for the solutions, using back pages as needed.

Problem 1. (20–pts, 5 pts each part) (a) (i) Define almost uniform convergence on the measure space $(X, \mathcal{F}, \mu)$.

(ii) Let f_n be a sequence of nonnegative measurable function converging almost uniformly to the nonnegative function f. Prove that $\sqrt{f_n}$ converges almost uniformly to $\sqrt{f}$.

(b) (i) Suppose f_n has the property that $\int_X |f_n| d\mu \to 0$. (i) Does it follow that $f_n \to 0$, a.e.? Justify your answer.

(ii) Does it follow that $f_n \to 0$, almost uniformly? Justify your answer.

Problem 2. **(20–pts)** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1 \leq p \leq \infty$ and q be its conjugate exponent. Suppose $f_n \to f$ in L^p and $g_n \to g$ in L^q. Prove that $f_n g_n \to fg$ in L^1.

Problem 3. **(20–pts)** Let $\{a_k\}$ be a sequence of positive numbers converging to infinity. Prove that the following limit exists

$$\lim_{k\to\infty} \int_0^\infty \frac{e^{-x}\cos x}{a_k x^2 + \frac{1}{a_k}}\, dx$$

and find it. Make sure to justify all steps.

Problem 4. **(20–pts)** Let $(X, \mathcal{F}, \mu)$ be σ-finite and f be measurable such that for all $\lambda > 0$,

$$\mu\{x : |f(x)| > \lambda\} \leq \frac{20}{\lambda^p}$$

where $1 < p < \infty$. Let q be the conjugate exponent of p. Prove that that there is a constant C depending only on p such that such

$$\int_E |f(x)| d\mu \leq C m(E)^{1/q},$$

for all measurable sets E with $0 < \mu(E) < \infty$. (The inequality holds trivially when $\mu(E) = 0$ or $\mu(E) = \infty$.)
(Hint: Recall $\int_E |f(x)| d\mu = \int_0^\infty ?\, d\lambda$ and "break it" at the right place!)

Problem 5. (20–pts) Suppose $f : [0,1] \to \mathbb{R}$ is of bounded variation with $V(f;0,1) = \alpha$. For any $\beta > 0$, set

$$A = \left\{ x \in (0,1) : \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} > \beta \right\}.$$

Prove that for any $0 < p < 1$, $m(A) \leq \frac{\alpha^p}{\beta^p}$, where m denotes the Lebesgue measure.

Problem 6. (20–pts, 10 pts each part) Let $f \in L^1(0,1)$ and for $x \in (0,1)$, define

$$h(x) = \int_x^1 \frac{1}{t} f(t)dt.$$

(i) Prove that h is continuous on $(0,1)$.

(ii) Show that

$$\int_0^1 h(t)dt = \int_0^1 f(t)dt.$$