REAL ANALYSIS QUALIFYING EXAM— AUGUST 2014

Student Identifier: ____

INSTRUCTIONS:
1) Justify your answers.
2) There is one problem per sheet of paper, where its solution should be written. Feel free to use the front and the back of the sheet, but please do not write the solution of one problem in the space reserved for another. If you need extra paper, please ask the proctor. Do not use your own paper.

1) (10 points) Discuss the convergence of

$$\sum_{N=0}^{\infty} \int_{(0,1)} \left(1 - x^p\right)^N \, dx, \quad p > 0,$$

for different values of p.

2) (15 points) Use the relationship between the Riemann and the Lebesgue integrals to compute

$$\lim_{N\to\infty} \sum_{j=1}^{2^N} \left(\sqrt{\frac{j}{2^N}} - \sqrt{\frac{j-1}{2^N}} \right) \frac{j-1}{2^N}.$$

3)(15 points) Let $K \in C([0,1] \times [0,1])$ (a continuous function) and let $f_n \in L^1([0,1])$, $n \in \mathbb{N}$, be such that

$$\int_{[0,1]} |f_n(x)|\, dx \leq 1.$$

Let

$$g_n(x) = \int_{[0,1]} K(x,y) f_n(y)\, dy.$$

Show that $\{g_n, n \in \mathbb{N}\}$ is a uniformly equicontinuous family in $C([0,1])$. Does $\{g_n\}$ have a pointwise convergent subsequence? Does $\{g_n\}$ have a uniformly convergent subsequence?

4) (20 points) Let $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Suppose that for every function $g \in C_c(\mathbb{R}^n)$, (continuous of compact support),

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq M||g||_{L^q(\mathbb{R}^n)}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Show that $||f||_{L^p(\mathbb{R}^n)} \leq M$.

5) (20 points) Let $(X, \mathcal{M}, \mu)$ be a measure space (μ positive). Let $f : X \longrightarrow \mathbb{R}$ be a measurable function. Show that

$$\int_X f^k \, d\mu = I, \text{ a constant, for all } k \in \mathbb{N}$$

if and only if $f = \chi_A$, for a measurable set $A \subset X$ with $\mu(A) < \infty$.

6) (20 points) Let $f, g \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Show that

$$\psi(x) = \int_{\mathbb{R}^n} f(x+y)g(y)\, dy$$

is a uniformly continuous function in $\mathbb{R}^n$.