MA 544 Real Analysis Qualifying Exam
August 2015

Instructions:

(1) Show your reasoning in all problems.

(2) There are six problems in this exam. A problem appears on each of the following pages. Use the space provided for the solutions, using back pages as needed. If you need extra paper, please ask the proctor.

(15 pts) 1. Let m denote Lebesgue measure on $\mathbb{R}$. Prove that the following limit exists

$$\lim_{n\to\infty}\int_{[0,1]} n(1+\frac{x}{n^2})^{-2}x^{-3/2}\sin(\frac{x}{n})\,dm$$

and find it. Justify your steps.

(15 pts) 2. Assume $1 \leq p < \infty$, $\{g_k\}$ and g are in $L^p(\mathbb{R}^n)$, and $g_k \to g$ in $L^p(\mathbb{R}^n)$. Prove that if $\{\lambda_k\} \subset \mathbb{R}^n$ and $\lambda_k \to 0$, then

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |g_k(x+\lambda_k) - g(x)|^p \, dm = 0,$$

where m denotes Lebesgue measure in $\mathbb{R}^n$.

(15 pts) 3. Assume f is a Lebesgue measurable function on $\mathbb{R}$ satisfying:

(i) there exists $p \in (1, \infty)$ such that $f \in L^p(I)$ for every bounded interval I in $\mathbb{R}$.

(ii) there exists $\theta \in (0, 1)$ such that

$$|\int_I f \, dm|^p \leq \theta \cdot (m(I))^{p-1} \cdot \int_I |f|^p \, dm$$

where m denotes Lebesgue measure in $\mathbb{R}$.

Prove that $f = 0$ almost everywhere in $\mathbb{R}$.

(15 pts) 4. Let m denote Lebesgue measure in $\mathbb{R}^n$. Assume E is a Lebesgue measurable set in $\mathbb{R}^n$ such that $m(E) < \infty$. For each $x \in \mathbb{R}^n$, let $x + E$ denote the translation of E by x:

$$x + E = \{y \in \mathbb{R}^n : y = x + z \text{ for some } z \in E\}.$$

Prove that the function defined by $g(x) = m([x + E] \cap E)$ for $x \in \mathbb{R}^n$ satisfies

$$\lim_{|x| \to \infty} g(x) = 0.$$

(20 pts) 5. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Assume $\{f_j\}$ and f are measurable functions from X into $[0, \infty)$ such that $f_j(x) \to f(x)$ pointwise almost everywhere on X as $j \to \infty$, and for some constant C,

$$\int_X f_j \, d\mu \leq C \text{ for all } j.$$

Prove that $\{\ln(1+f_j(x))\}$ and $\ln(1+f(x))$ are measurable functions on X and $\ln(1+f_j)$ converges to $\ln(1+f)$ in $L^1(X, \mathcal{M}, \mu)$.

(20 pts) 6. Assume f is a real-valued absolutely continuous function on $[0,1]$, $f(0) = 0$, $f' \in L^p([0,1])$ for some $1 < p < \infty$, and $g \in L^q([0,1])$ with respect to Lebesgue measure m, where q is the conjugate exponent of p. Prove that

$$\int_{[0,1]} |fg| \, dm \leq (\frac{1}{p})^{(\frac{1}{p})} \cdot \|f'\|_p \cdot \|g\|_q.$$