MA 54400 - Qualifying Exam

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Problem	Score	Max. pts.
1		20
2		25
3		25
4		30
Total		100

In order to receive full credit, you need to show your work and justify your arguments.

1. Recall that the distance between two disjoint, nonempty sets $S, T \subset \mathbb{R}$ is defined as
$$d(S,T) = \inf\{|s-t| \mid s \in S,\ t \in T\}.$$
Assume that $d(S,T) > 0$. Show that $|S \cup T|_e = |S|_e + |T|_e$. (Here $|E|_e$ denotes the outer Lebesgue measure of the set E).

2. Prove that, if $0 < \varepsilon < 1$, there is no measurable set $E \subset \mathbb{R}$ such that

$$\varepsilon < \frac{|E \cap I|}{|I|} < 1 - \varepsilon$$

for every interval $I \subset \mathbb{R}$.

3. Suppose that $p > 0$, $E \subset \mathbb{R}$ with $|E| < \infty$, and that f is a measurable function on E. Show that if

$$|\{x \in E \mid |f(x) > t\}| = O(t^{-p}) \quad \text{as } t \to +\infty,$$

then $f \in L^{p-\varepsilon}(E)$ for any $\varepsilon \in (0, p)$.

(*Hint:* Recall that $g(t) = O(h(t))$ as $t \to +\infty$ iff there exist a positive real number L and a real number t_0 such that $|g(t)| \leq L|h(t)|$ for all $t \geq t_0$. Use the distribution function to compute the integral in question.)

4. Let $\{f_n\}$ be a sequence of functions in $L^2(\mathbb{R})$, and let $f \in L^2(\mathbb{R})$. Assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n g \, dx \;=\; \int_{\mathbb{R}} fg \, dx$$

for all $g \in L^2(\mathbb{R})$.

1. Show that
$$\|f\|_{L^2(\mathbb{R})} \le \liminf_{n\to\infty} \|f_n\|_{L^2(\mathbb{R})}.$$
2. Suppose, in addition, that
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n^2 \, dx \;=\; \int_{\mathbb{R}} f^2 \, dx.$$
Prove that $\lim_{n\to\infty} \|f_n - f\|_{L^2(\mathbb{R})} = 0$.
3. **(Extra credit: 15 points)** Give an example where the inequality in Part 1. is strict, with the right-hand side being finite.