MA 54400 QUALIFIER, 1/4/2016

Each problem is worth 5 points. Make sure that you justify your answers. In problems 4 and 5 a measure space $(\Omega, \mathcal{A}, \mu)$ is given.

Notes, books, crib sheets, and electronic devices are not allowed. You need not copy down the formulation of the problems.

1. Given a bounded function $f : \mathbb{R} \to \mathbb{R}$, let

$$g(x) = \limsup_{y \to x} f(y) \ \big(= \lim_{r \to 0^+} \sup\{f(y) \ : \ 0 < |x-y| < r\} \ \big).$$

Prove that g is upper semicontinuous.

2. A subset $X \subset \mathbb{R}$ is called an F_σ–set if it can be written as the union of countably many closed subsets of $\mathbb{R}$. If $h : \mathbb{R} \to \mathbb{R}$ is continuous, and $X \subset \mathbb{R}$ is F_σ, prove that $h(X)$ is also F_σ.

3. Consider a function $\phi : \mathbb{R} \to \mathbb{R}$ with the property that whenever a sequence $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$ converges uniformly, so does the sequence $\phi \circ f_n$. Show that this property is equivalent to ϕ being uniformly continuous.

4. Suppose $\mu(\Omega) < \infty$, and $F : \Omega \to \mathbb{R}$ is bounded and measurable. Prove that

$$\int_\Omega F \ d\mu = \sup \Big\{ \int_\Omega G \ d\mu \ : \ G \le F \text{ is simple} \Big\}.$$

5. Let $1 \le a < b < \infty$ and $\psi \in L^a(\Omega, \mathcal{A}, \mu) \cap L^b(\Omega, \mathcal{A}, \mu)$. Denoting L^p–norm by $|| \cdot ||_p$, prove that the function

$$[a, b] \ni p \mapsto ||\psi||_p \in \mathbb{R}$$

is continuous.

6. Consider functions $u, v : [a, b] \to \mathbb{R}$, u continuous, v absolutely continuous. Suppose that at almost every $x \in [a, b]$ the derivative of v exists and equals $u(x)$. Prove that then v must be differentiable everywhere, and $v' = u$ on all of $[a, b]$.

7. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and suppose $E \subset [a, b]$ has measure 0. Prove that $f(E)$ also has measure 0.