MA 544 Qualifying Exam

Name:__________________________________

a) Legibly print your name above.

b) Do not open this test booklet until you are directed to do so.

c) You will have 120 min. to complete the exam. Budget your time wisely!

d) This test is closed book and closed notes. You may not use a calculator during this test.

e) Throughout the test, **show your work so that your reasoning is clear**.

f) If you need extra room, use the back of the pages. Just make sure I can follow your work.

Problem	Points	Grade
1	20	
2	10	
3	20	
4	20	
5	15	
6	15	
Total	100	

1 (20 pts)**.** Suppose that μ is a finite Borel measure on $[0, \infty)$. Prove that $\int e^{\alpha x}\, d\mu(x) < \infty$ for some $\alpha > 0$ **if and only if** there exist $c, C > 0$ such that $\mu([t, \infty)) \leq Ce^{-ct}$ for all $t > 0$.

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2 (10 pts). Let $\{f_k\}_{k\geq 1}$ and f be Lebesgue measurable functions on $\mathbb{R}^n$ such that $f_k \overset{m}{\to} f$ (note that $\overset{m}{\to}$ denotes convergence in measure). Prove that if the functions $\{f_k\}_k$ are uniformly bounded (that is $|f_k(x)| \leq M < \infty$ for all $x \in \mathbb{R}^n$ and $k \geq 1$) then $\phi(f_k) \overset{m}{\to} \phi(f)$ for every continuous function $\phi : \mathbb{R} \to \mathbb{R}$.

3 (20 pts)**.** Compute the following limit

$$\lim_{n\to\infty} \sum_{k=0}^{\lfloor rn \rfloor} \left(1 - \frac{k}{n}\right)^n$$

for any $0 < r < 2$. Make sure to fully justify all of your calculations.
Hint: it may be helpful to first consider the case $0 < r \leq 1$.

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4 (20 pts)**.** Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Prove that if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then

$$\lim_{|x|\to\infty} (f * g)(x) = \lim_{|x|\to\infty} \int_{\mathbb{R}} f(x-y)g(y)\,dy = 0.$$

Hint: divide the integral for $(f * g)(x)$ into two parts and analyze each part separately.

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5 (15 pts)**.** For any absolutely continuous function f on $[0, 1]$, let

$$J(f) = \int_0^1 f'(x)^2 \, dx.$$

Let $\mathcal{A}$ be the set of absolutely continuous functions, and for any $t > 0$ let

$$\mathcal{A}_t = \left\{ f \in \mathcal{A} : \ f(0) = 0 \text{ and } \sup_{x \in [0,1]} f(x) \geq t \right\}.$$

Prove that

$$\inf_{f \in \mathcal{A}_t} J(f) = t^2.$$

Hint: first prove that $J(f) \geq t^2$ *for all* $f \in \mathcal{A}_t$*.*

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6 (15 pts)**.** Compute

$$\lim_{n\to\infty} \iint_{(0,\infty)^2} \frac{n}{x} \sin\left(\frac{x}{ny}\right) e^{-\frac{x}{y}-y}\,(dx\,dy).$$

Make sure to fully justify your computations.

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